



A T M E
College of Engineering



Finite Element Method BME701

Module-2 One Dimensional Element

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Module - 2

One Dimensional Element:

Formulation of a linear bar element, Shape Functions Polynomial, The Potential Energy Approach, derivation of stiffness matrix, Properties of stiffness matrix, Assembly of Global Stiffness Matrix and Load Vector, Boundary conditions-elimination method and penalty method. Numerical Problems on straight and stepped bars. (Problems with 2 elements only).

Properties of Stiffness matrix

- If there are 'n' number of nodes, the global stiffness matrix is **(n×n)** provided one DOF at each node.
- Stiffness matrix is a **symmetric** matrix.
- Stiffness matrix is a **banded** matrix.
- The **diagonal elements** in a matrix are always positive.
- The sum of elements in any column must be equal to **Zero**.
- The determinant of the matrix will be always **Zero**.

Stiffness Matrix:

The primary characteristics of a finite element are embodied in the element **stiffness matrix**.

For a structural finite element, the stiffness matrix contains the geometric and material behavior information that indicates the resistance of the element to deformation when subjected to loading.

For finite elements used in non-structural analyses, such as fluid flow and heat transfer, The term stiffness matrix is also used, since **the matrix represents the resistance of the element to change when subjected to external influences**.

Governing displacement equation of a system is given by

$$[K_e]\{Q\} = \{F\}$$

Where,

$[K_e]$ is the global stiffness matrix., $[Q]$ is the displacement vector, $[F]$ is the force vector

Derivation of element stiffness matrix for 1D bar element / rod element:

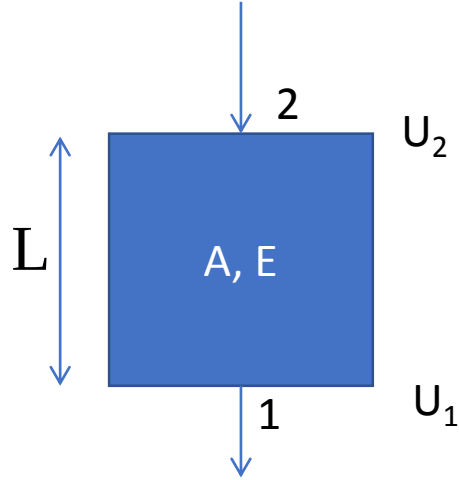


Fig shows a bar subjected to axial load at both ends, the bar is taken as 1 D bar element and each node as one DOF.

The total energy = Sum of strain energy + work potential

$$\Pi = \frac{AE}{2} \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx - f_1 u_1 - f_2 u_2 \text{ -----}^*$$

Displacement model

$$U = a_0 + a_1 x \text{----- } 1$$

Applying Boundary Condition

$$@x = 0, U = U_1$$

$$U_1 = a_0 + a_1(0)$$

$$U_1 = a_0 \text{----- } a$$

$$@x = L, U = U_2$$

$$U_2 = a_0 + a_1(L)$$

$$U_2 = U_1 + a_1(L)$$

$$U_2 - U_1 = a_1(L)$$

$$a_1 = \frac{U_2 - U_1}{L} \text{----- } b$$

Substitute values of a and b in equation 1

$$U = a_0 + a_1 x$$

$$U = U_1 + \left(\frac{U_2 - U_1}{L} \right) x \text{-----2}$$

Differentiate equation 2 w.r.t x

$$\frac{\partial U}{\partial x} = \frac{U_2 - U_1}{L}$$

Squaring on both side

$$\left(\frac{\partial U}{\partial x} \right)^2 = \left(\frac{U_2 - U_1}{L} \right)^2$$

$$\left(\frac{\partial U}{\partial x} \right)^2 = \left(\frac{U_2^2 + U_1^2 - 2U_1U_2}{L^2} \right) \text{-----3}$$

Substitute equation 3 in *

$$\Pi = \frac{AE}{2} \int_0^L \left(\frac{U_2^2 + U_1^2 - 2U_1U_2}{L^2} \right)^2 dx - f_1 u_1 - f_2 u_2$$

On solving,

$$\Pi = \frac{AE}{2} \left[\left(\frac{U_2^2 + U_1^2 - 2U_1U_2}{L^2} \right) x \right]_0^L - f_1u_1 - f_2u_2$$

$$\Pi = \frac{AE}{2} \left[\left(\frac{U_2^2 + U_1^2 - 2U_1U_2}{L^2} \right) (L - 0) \right] - f_1u_1 - f_2u_2$$

$$\Pi = \frac{AE}{2} \left[\left(\frac{U_2^2 + U_1^2 - 2U_1U_2}{L} \right) \right] - f_1u_1 - f_2u_2 \text{ ----- 4}$$

For maximum condition, differentiate equation 4 w.r.t dU_1 , dU_2 & equate to Zero

$$\frac{d\pi}{dU_1} = \frac{AE}{2L} [2U_1 - 2U_2] - f_1 = 0 \text{ ---- 5}$$

$$\frac{d\pi}{dU_2} = \frac{AE}{2L} [2U_2 - 2U_1] - f_2 = 0 \text{ ---- 6}$$

On simplification,

$$\frac{AE}{L}[U_1 - U_2] = f_1 \text{ ----- } 7$$

$$\frac{AE}{L}[-U_1 + U_2] = f_2 \text{ ----- } 8$$

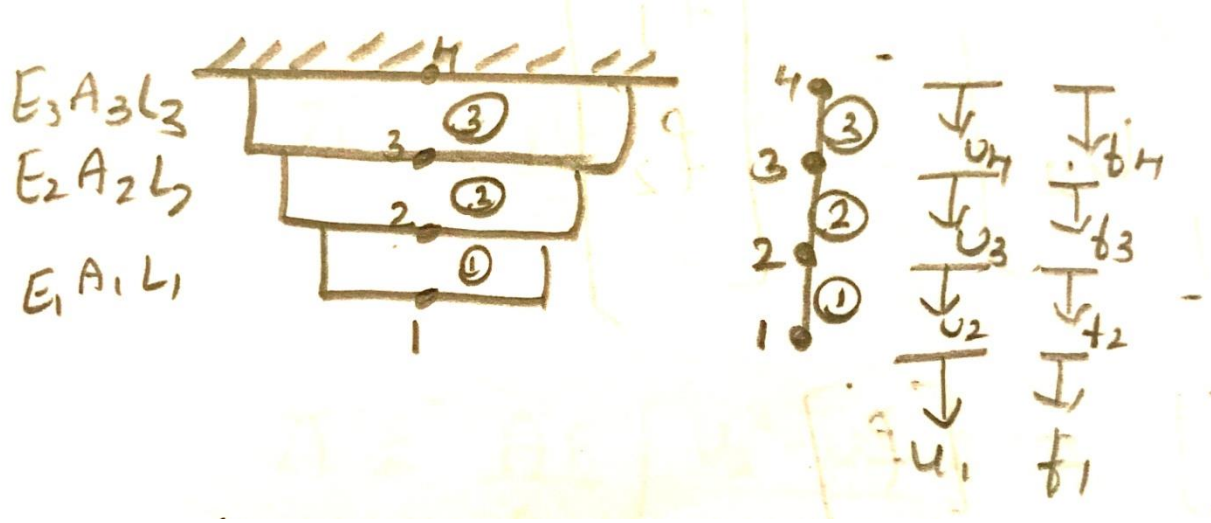
Substitute equation 7 & 8 in matrix form

$$\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$[K][U] = [f] \quad \text{where, Stiffness matrix } [K] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

Assembling of Element Stiffness Matrix into global stiffness matrix



In general,

$$[K] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

For element -1:

$$\begin{matrix} & 1 & 2 \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} \frac{A_1 E_1}{L_1} & -\frac{A_1 E_1}{L_1} \\ -\frac{A_1 E_1}{L_1} & \frac{A_1 E_1}{L_1} \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \end{matrix}$$

For element -2:

$$\begin{matrix} & 2 & 3 \\ \begin{matrix} 2 \\ 3 \end{matrix} & \begin{bmatrix} \frac{A_2 E_2}{L_2} & -\frac{A_2 E_2}{L_2} \\ -\frac{A_2 E_2}{L_2} & \frac{A_2 E_2}{L_2} \end{bmatrix} & \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} \end{matrix}$$

For element -3:

$$\begin{matrix} & 3 & 4 \\ \begin{matrix} 3 \\ 4 \end{matrix} & \begin{bmatrix} \frac{A_3 E_3}{L_3} & -\frac{A_3 E_3}{L_3} \\ -\frac{A_3 E_3}{L_3} & \frac{A_3 E_3}{L_3} \end{bmatrix} & \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \end{matrix}$$

Global Stiffness Matrix:

$$\begin{matrix} & 1 & 2 & 3 & 4 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \frac{A_1 E_1}{L_1} & -\frac{A_1 E_1}{L_1} & 0 & 0 \\ -\frac{A_1 E_1}{L_1} & \frac{A_1 E_1}{L_1} + \frac{A_2 E_2}{L_2} & -\frac{A_2 E_2}{L_2} & 0 \\ 0 & -\frac{A_2 E_2}{L_2} & \frac{A_2 E_2}{L_2} + \frac{A_3 E_3}{L_3} & -\frac{A_3 E_3}{L_3} \\ 0 & 0 & -\frac{A_3 E_3}{L_3} & \frac{A_3 E_3}{L_3} \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \end{matrix}$$

Analysis of 1D Bar:

Steps involved in solving of 1D bar problem:

1. Finite Element model of a continuum/problem (with nodes & elements)
2. Determination of element stiffness matrix for individual element

$$[K] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

3. Determination of global stiffness matrix
4. Determination of global force vector and displacement vector [q] and [f]
5. Equilibrium equation $[K][q] = [f]$
6. Applying boundary conditions
 1. By Elimination method
 2. By penalty method

7. Determination of stress and strain in each element

a. Stress:

$$\sigma = E B q$$

$$\sigma = E \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} [q]$$

b. Strain:

$$\sigma = E * \varepsilon$$

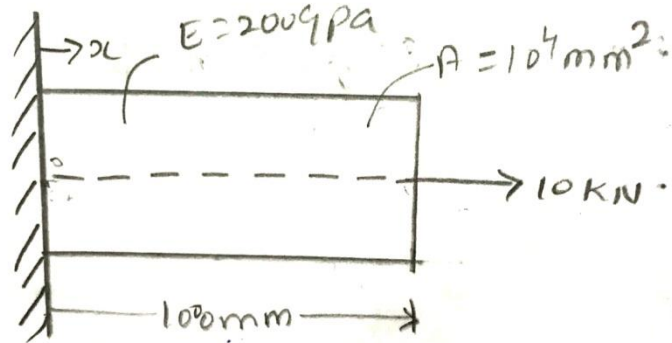
8. Determination of reaction at the support

$$[R] = [K][q] - [f]$$

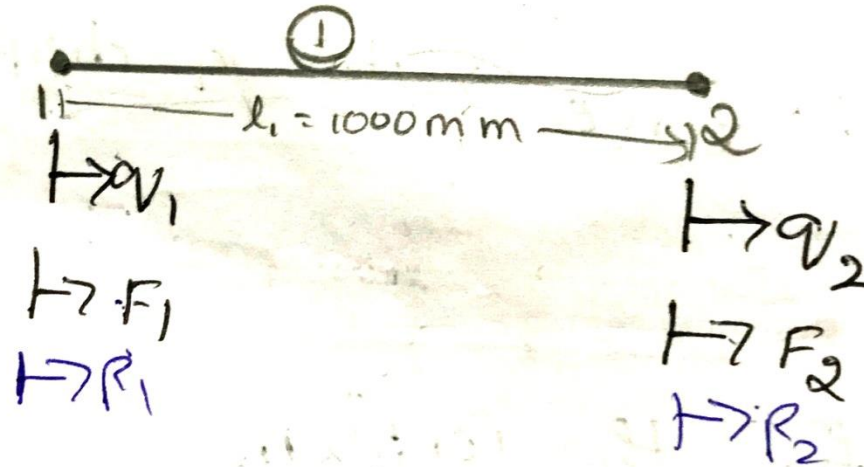
Numerical on 1D bar:

- The fig shows 1D bar element subjected to an axial load. Taking it as a single element determine, a. Nodal displacement, b. Stress in each element, c. reaction at the support.

Take $E_1 = 200\text{GPa} = 2 \times 10^5\text{Mpa}$, $A_1 = 10^4\text{mm}^2$, $l_1 = 1000\text{mm}$, $F_2 = 10\text{kN} = 10^4\text{N}$



- Finite element model:



2. Element stiffness matrix for each element:

In general,

$$[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{10^4 * 2 * 10^5}{10^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 2 * 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^6 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^6 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

5. Equilibrium Equation:

$$[K][q]=[F]$$

$$10^6 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

6. Applying Boundary condition:

By elimination method:

Since node 1 is fixed, $q_1 = 0$

Eliminate 1st row and 1st column

$$10^6 \begin{bmatrix} \cancel{2} & \cancel{-2} \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \cancel{q_1} \\ q_2 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ 10^4 \end{bmatrix}$$

$$10^6 * 2 * q_2 = 10^4$$

$$q_2 = 5 * 10^{-3} \text{mm} = 0.005 \text{mm}$$

7. To find stress and strain:

Stress: $\sigma = E \cdot B \cdot q$

$$\sigma_1 = E_1 \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_1 = 2 \cdot 10^5 * \frac{1}{1000} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.005 \end{bmatrix}$$

$$\sigma_1 = 200 * 0.005$$

$$\begin{bmatrix} -200 & 200 \end{bmatrix} \begin{bmatrix} 0 \\ 0.005 \end{bmatrix}$$

$$\sigma_1 = 1 \text{ N/mm}^2$$

Strain: $\sigma_1 = E_1 * \varepsilon_1$

$$\varepsilon_1 = \sigma_1 / E_1$$

$$\varepsilon_1 = 1/2 * 10^5$$

$$\varepsilon_1 = 5 * 10^{-6}$$

8. Determination of reaction at the support:

$$[R] = [K][q] - [F]$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = 10^6 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.005 \end{bmatrix} - \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = 10^6 \begin{bmatrix} 0 + (-0.01) \\ -0 + 0.01 \end{bmatrix} - \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = 10^6 \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix} - \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} -10^4 \\ 10^4 \end{bmatrix} - \begin{bmatrix} 0 \\ 10^4 \end{bmatrix}$$

$$R_1 = -10^4 - 0 = -10^4 = -10 \times 10^3 \text{ N} = -10 \text{ kN}$$

$$R_2 = 10^4 - 10^4 = 0 \text{ N}$$

1. Nodal displacement:

$$q_1 = 0$$

$$q_2 = 0.005 \text{ mm}$$

2. Stress:

$$\sigma_1 = 1 \text{ Mpa}$$

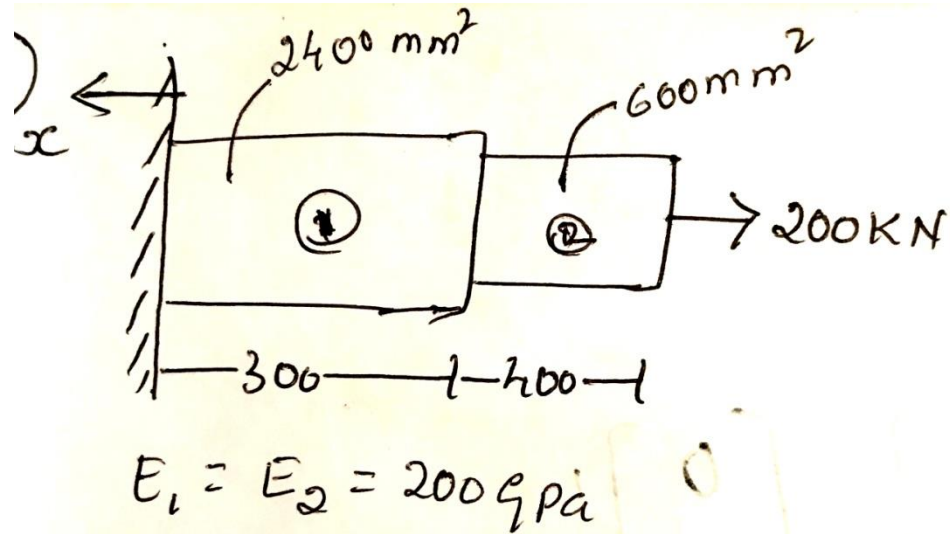
$$\varepsilon_1 = 5 \times 10^{-6} \text{ Mpa}$$

3. Reaction force:

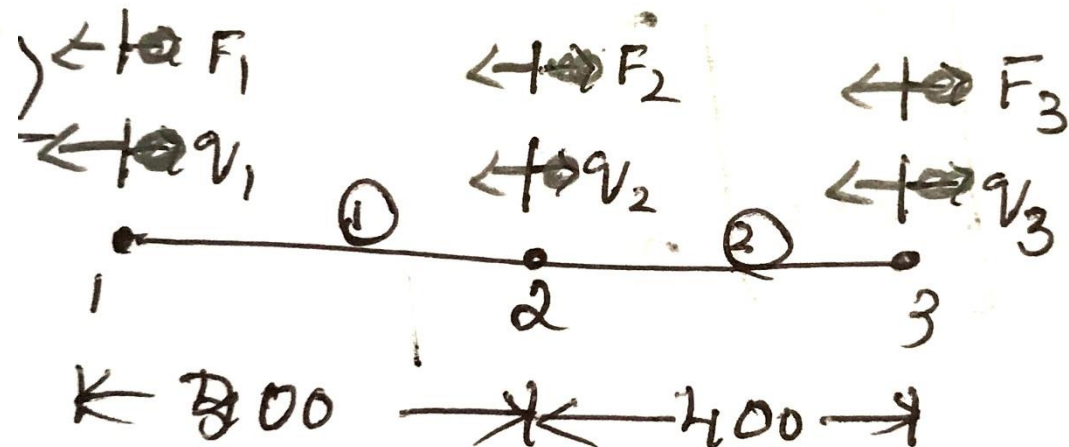
$$R_1 = -10 \text{ kN}$$

$$R_2 = 0$$

2. The fig shows 1D Stepped bar element subjected to an axial load. Determine, a. Nodal displacement, b. Stress in each element, c. reaction at the support.



1. Finite element model:



2. Element stiffness matrix for each element:

In general, $[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{2400 * 2 * 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^6 \begin{bmatrix} 1.6 & -1.6 \\ -1.6 & 1.6 \end{bmatrix}$$

Element 2:

$$[K_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = \frac{600 * 2 * 10^5}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^6 \begin{bmatrix} 0.3 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^6 \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 1.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \times 10^5 \end{bmatrix}$$

5. Equilibrium Equation:

$$[K][q] = [F]$$

$$10^6 \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 1.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \times 10^5 \end{bmatrix}$$

6. Applying Boundary condition:

By elimination method:

Since node 1 is fixed, $q_1 = 0$

Eliminate 1st row and 1st column

$$10^6 \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 1.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \times 10^5 \end{bmatrix}$$

$$10^6 \begin{bmatrix} 1.9 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \times 10^5 \end{bmatrix}$$

On simplification:

$$1.9q_2 10^6 - 0.3q_3 10^6 = 0$$

$$-0.3q_2 10^6 + 0.3q_3 10^6 = -2 * 10^5$$

$$q_2 = -0.125 \text{mm}$$

$$q_3 = -0.7916 \text{mm}$$

7. To find stress and strain:

Stress: $\sigma = EBq$

$$\sigma_1 = E_1 \frac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_1 = 2 * 10^5 * \frac{1}{300} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.125 \end{bmatrix}$$

$$\sigma_1 = -83.33 \text{MPa}$$

$$\sigma_2 = E_2 \frac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

$$\sigma_2 = 2 * 10^5 * \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -0.125 \\ -0.7916 \end{bmatrix}$$

$$\sigma_2 = -333.33 \text{MPa}$$

8. Determination of reaction at the support:

$$[R] = [K][q] - [F]$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^6 \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 1.9 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0 \\ -0.125 \\ -0.791 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -2 \times 10^5 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^6 \begin{bmatrix} 0 + 0.2 - 0 \\ 0 - 0.237 + 0.237 \\ 0 + 0.0375 - 0.2373 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -2 \times 10^5 \end{bmatrix}$$

$$R_1 = 10^6 * 0.2 - 0 = 2 \times 10^5 \text{ N}$$

$$R_2 = 10^6(0) - 0 = 0 \text{ N}$$

$$R_3 = 10^6 * -0.1998 - (-2 \times 10^5) = 200 \text{ N}$$

1. Nodal displacement:

$$q_1 = 0$$

$$q_2 = -0.125 \text{ mm}$$

$$q_3 = -0.791 \text{ mm}$$

2. Stress:

$$\sigma_1 = -83.33 \text{ Mpa}$$

$$\sigma_2 = -333.33 \text{ Mpa}$$

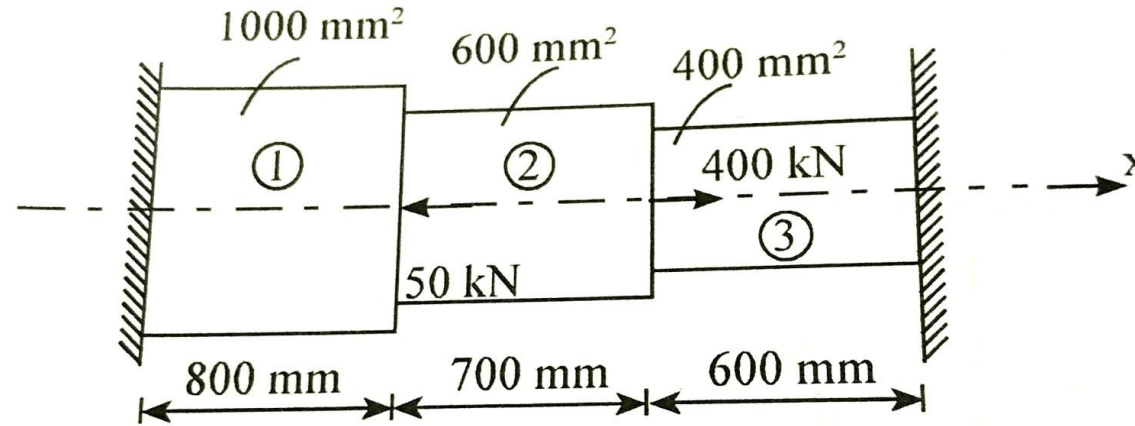
3. Reaction force:

$$R_1 = 2 \times 10^5 \text{ N}$$

$$R_2 = 0$$

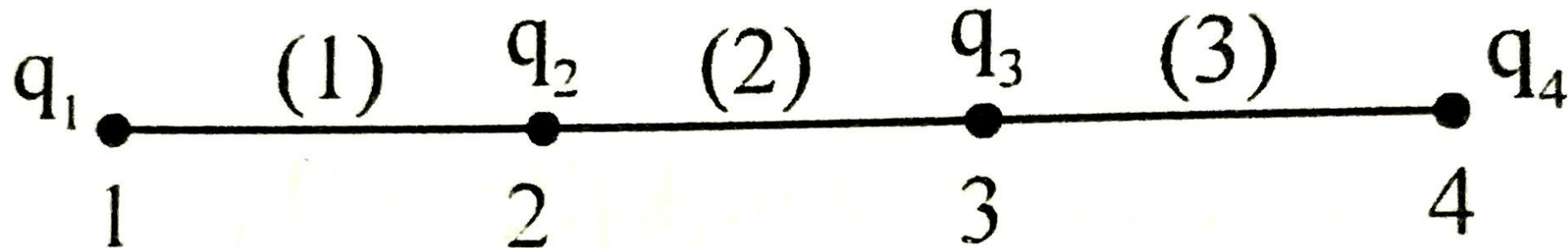
$$R_3 = 200 \text{ N}$$

3. The fig shows 1D bar element subjected to an axial load. determine, a. Nodal displacement, b. Stress in each element, c. reaction at the support. Take $E = 2 \times 10^5 \text{ MPa}$



Small correction: @node 3 : load = 100kN

1. Finite element model:



2. Element stiffness matrix for each element:

In general, $[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{1000 * 2 * 10^5}{800} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^5 \begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Element 2:

$$[K_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = \frac{600 * 2 * 10^5}{700} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^5 \begin{bmatrix} 1.714 & -1.714 \\ -1.714 & 1.714 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Element 3:

$$[K_3] = \frac{A_3 E_3}{L_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_3] = \frac{400 * 2 * 10^5}{600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^5 \begin{bmatrix} 1.33 & -1.33 \\ -1.33 & 1.33 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^5 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2.5 & -2.5 & 0 & 0 \\ -2.5 & 4.214 & -1.714 & 0 \\ 0 & -1.714 & 3.044 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -50*10^3 \\ 100*10^3 \\ 0 \end{bmatrix}$$

5. Equilibrium equation:

$$[K][q]=[F]$$

$$10^5 \begin{bmatrix} 2.5 & -2.5 & 0 & 0 \\ -2.5 & 4.214 & -1.714 & 0 \\ 0 & -1.714 & 3.044 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -50*10^3 \\ 100*10^3 \\ 0 \end{bmatrix}$$

6. Applying Boundary condition:

By elimination method:

Since node 1 and 4 is fixed,

$$q_1 = 0, q_4 = 0$$

Eliminate 1st, 4th row and 1st, 4th column

$$10^5 \begin{bmatrix} 2.5 & -2.5 & 0 & 0 \\ -2.5 & 4.214 & -1.714 & 0 \\ 0 & -1.714 & 3.044 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -50*10^3 \\ 100*10^3 \\ 0 \end{bmatrix}$$

$$10^5 \begin{bmatrix} 4.214 & -1.714 \\ -1.714 & 3.044 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -50*10^3 \\ 100*10^3 \end{bmatrix}$$

On simplification

$$4.214q_2 10^5 - 1.714q_3 10^5 = -50 \cdot 10^3$$

$$-1.714q_2 10^5 + 3.044q_3 10^5 = 100 \cdot 10^3$$

$$q_2 = 0.0194 \text{ mm}$$

$$q_3 = 0.339 \text{ mm}$$

7. To find stress and strain:

Stress: $\sigma = E \epsilon$

$$\sigma_1 = E_1 \frac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_1 = 2 \cdot 10^5 * \frac{1}{800} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.0194 \end{bmatrix}$$

$$\sigma_1 = 4.85 \text{ MPa}$$

$$\sigma_2 = E_2 \frac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

$$\sigma_2 = 2 \cdot 10^5 * \frac{1}{700} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.0194 \\ 0.3394 \end{bmatrix}$$

$$\sigma_2 = 91.43 \text{ MPa}$$

$$\sigma_3 = E_3 \frac{1}{l_3} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix}$$

$$\sigma_3 = 2 \cdot 10^5 * \frac{1}{600} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.3394 \\ 0 \end{bmatrix}$$

$$\sigma_3 = -113.13 \text{ MPa}$$

8. Determination of reaction at the support:

$$[R] = [K][q] - [F]$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = 10^5 \begin{bmatrix} 2.5 & -2.5 & 0 & 0 \\ -2.5 & 4.214 & -1.714 & 0 \\ 0 & -1.714 & 3.044 & -1.33 \\ 0 & 0 & -1.33 & 1.33 \end{bmatrix} \begin{bmatrix} 0 \\ 0.0194 \\ 0.339 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -50 \times 10^3 \\ 100 \times 10^3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = 10^5 \begin{bmatrix} 0 - 0.2285 + 0 + 0 \\ 0 \\ 0 \\ 0 + 0 - 0.4508 + 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -50 \times 10^3 \\ 100 \times 10^3 \\ 0 \end{bmatrix}$$

$$R_1 = -22850 \text{ N}$$

$$R_4 = -45080 \text{ N}$$

1. Nodal displacement:

$$q_1 = 0$$

$$q_2 = 0.0194 \text{ mm}$$

$$q_3 = 0.339 \text{ mm}$$

$$q_4 = 0$$

2. Stress:

$$\sigma_1 = 4.85 \text{ MPa}$$

$$\sigma_2 = 91.43 \text{ MPa}$$

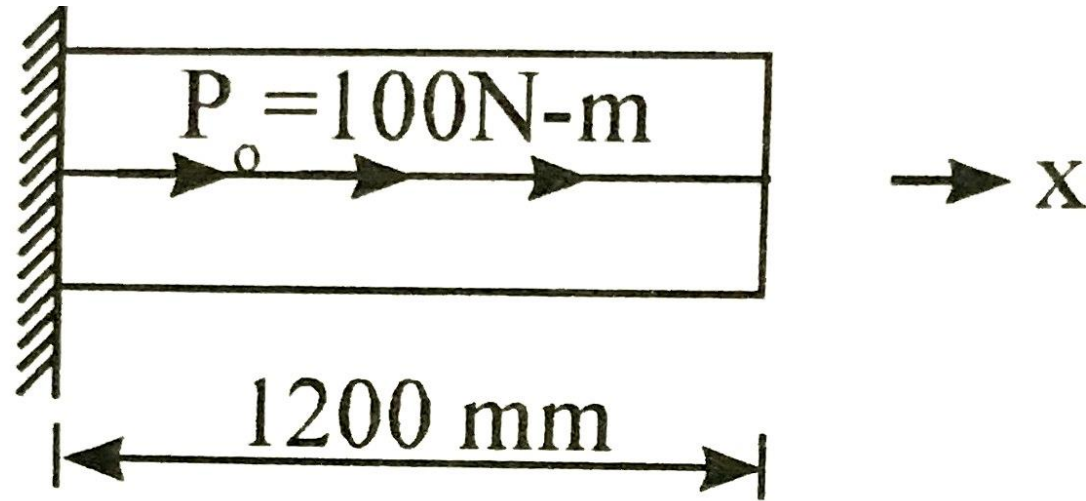
$$\sigma_3 = -113.13 \text{ MPa}$$

3. Reaction force:

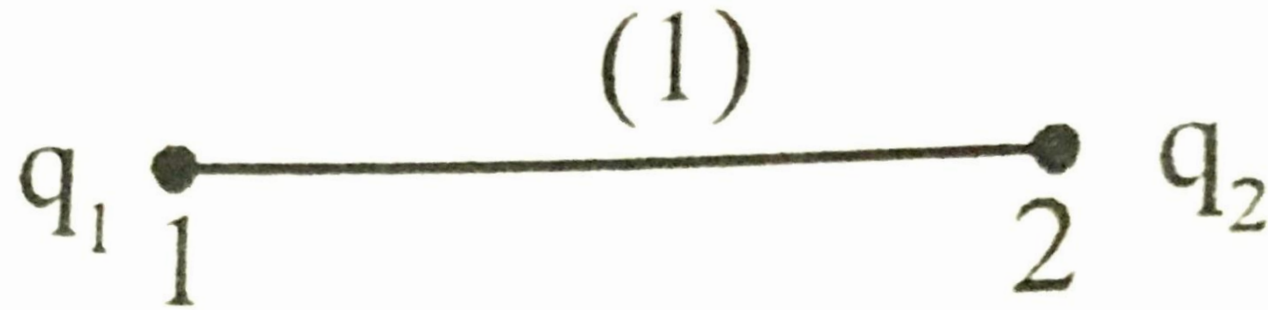
$$R_1 = -22850 \text{ N}$$

$$R_4 = -45080 \text{ N}$$

4. Figure shows a bar subjected to uniform distributed load P_0 . Take $E = 70\text{GPa}$, $A = 10^4\text{mm}^2$. Determine the a. nodal displacement, b. Stress in the element.



1. Finite element model:



2. Element stiffness matrix for each element:

In general,

$$[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{10^4 * 0.7 * 10^5}{1200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 5.833 * 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^5 \begin{bmatrix} 5.833 & -5.833 \\ -5.833 & 5.833 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^5 \begin{bmatrix} 5.833 & -5.833 \\ -5.833 & 5.833 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad [F] = \frac{P_0 l}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{100 * 10^3 * 1200}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 60 * 10^6 \\ 60 * 10^6 \end{bmatrix}$$

5. Equilibrium Equation:

$$[K][q]=[F]$$

$$10^5 \begin{bmatrix} 5.833 & -5.833 \\ -5.833 & 5.833 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 60 \cdot 10^6 \\ 60 \cdot 10^6 \end{bmatrix}$$

6. Applying Boundary condition:

By elimination method:

Since node 1 is fixed, $q_1 = 0$

Eliminate 1st row and 1st column

$$10^5 \begin{bmatrix} 5.833 & -5.833 \\ -5.833 & 5.833 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 60 \cdot 10^6 \\ 60 \cdot 10^6 \end{bmatrix}$$

$$5.833 \cdot 10^5 \cdot q_2 = 60 \cdot 10^6$$

$$\mathbf{q_2 = 102.86 \text{ mm}}$$

7. To find stress and strain:

Stress: $\sigma = E * B * q$

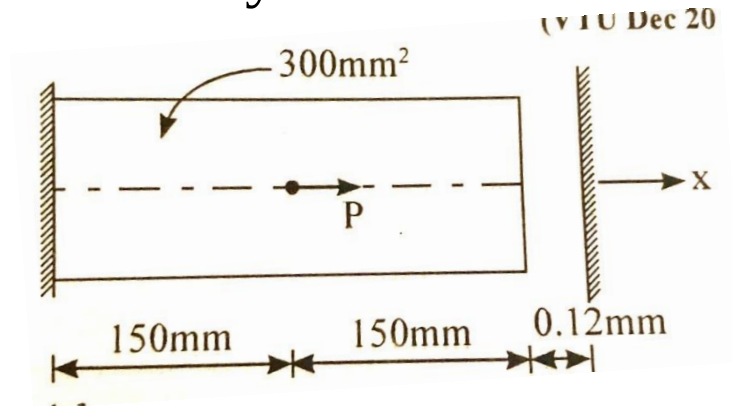
$$\sigma_1 = E_1 \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_1 = 0.7 * 10^5 * \frac{1}{1200} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 102.86 \end{bmatrix}$$

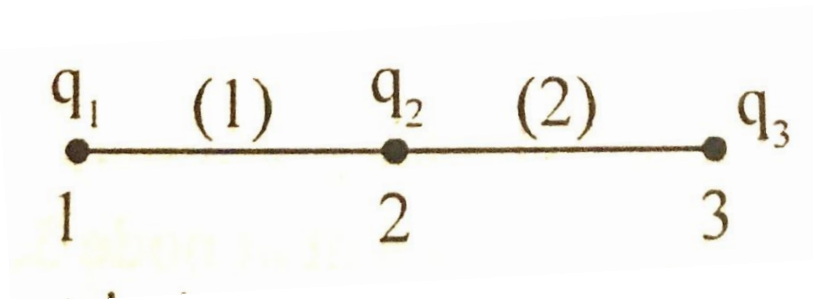
$$\sigma_1 = 58.33 * 102.86$$

$$\sigma_1 = \mathbf{6000 \text{ N/mm}^2}$$

5. A bar is having uniform cross sectional area of 300mm^2 and is subjected to a load $P = 600\text{kN}$ as shown in fig. determine the displacement field, stress and support reaction in the bar. Consider two element and rise elimination method to handle boundary conditions. Take $E = 200\text{GPa}$.



1. Finite Element model:



2. Element stiffness matrix for each element:

In general, $[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{300 \cdot 2 \cdot 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^5 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

Element 2:

$$[K_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = \frac{300 \cdot 2 \cdot 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^5 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} F_1 - K_{13}a_3 \\ F_2 - K_{23}a_3 \\ F_3 - K_{33}a_3 \end{bmatrix} = \begin{bmatrix} 0 - 0*0.12 \\ 600*10^3 - (-4*10^5)*0.12 \\ 0 - 4*10^5*0.12 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.48*10^5 \\ 0.48*10^5 \end{bmatrix}$$

Where a_3 is maximum displacement at node 3 i.e., $a_3 = 0.12\text{mm}$

5. Equilibrium Equation:

$$[K][q] = [F]$$

$$10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.48*10^5 \\ 0.48*10^5 \end{bmatrix}$$

6. Applying Boundary condition:

By elimination method:

Since node 1 is fixed, $q_1 = 0$ and $q_3 = 0.12$

Eliminate 1st, 3rd row and 1st, 3rd column

$$10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6.48*10^5 \\ 0.48*10^5 \end{bmatrix}$$

On simplification:

$$8*10^5 q_2 = 6.48*10^5$$

$$q_2 = 0.81\text{mm}$$

7. To find stress and strain:

Stress: $\sigma = EBq$

$$\sigma_1 = E_1 \frac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_1 = 2*10^5 * \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.81 \end{bmatrix}$$

$$\sigma_1 = 1080 \text{ MPa}$$

$$\sigma_2 = E_2 \frac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

$$\sigma_2 = 2*10^5 * \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.81 \\ 0.12 \end{bmatrix}$$

$$\sigma_2 = -920\text{MPa}$$

8. Determination of reaction at the support:

$$[R] = [K][q] - [F]$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0.81 \\ 0.12 \end{bmatrix} - \begin{bmatrix} 0 \\ 600 \times 10^3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} -3.24 \\ 6 \\ -2.76 \end{bmatrix} - \begin{bmatrix} 0 \\ 600 \times 10^3 \\ 0 \end{bmatrix}$$

$$R_1 = 10^5 \times -3.24 - 0 = -324000 \text{ N}$$

$$R_3 = 10^5 \times -2.76 - (0) = -276000 \text{ N}$$

1. Nodal displacement:

$$q_1 = 0$$

$$q_2 = 0.81 \text{ mm}$$

$$q_3 = 0.12 \text{ mm}$$

2. Stress:

$$\sigma_1 = 1080 \text{ Mpa}$$

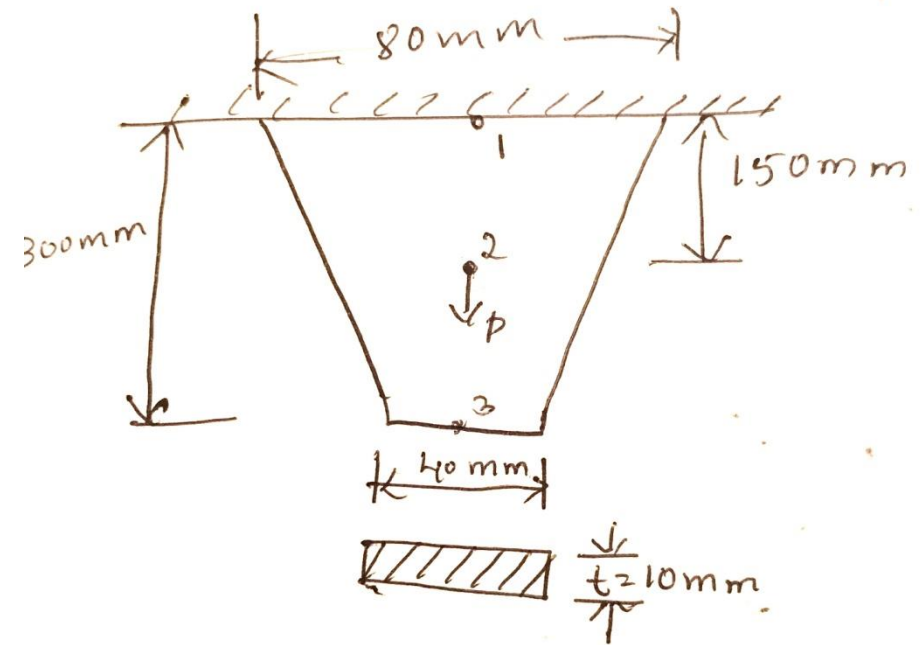
$$\sigma_2 = -920 \text{ Mpa}$$

3. Reaction force:

$$R_1 = -324000 \text{ N}$$

$$R_3 = -276000 \text{ N}$$

6. Figure shows a thin plate of uniform thickness of 10mm, young's modulus $E = 200\text{Gpa}$, weight density of the plate is 7800 kg/m^3 , in addition to its weight subjected to a point load of 1kN at its mid point. Determine nodal displacement, stress and reaction support.



Given,

$$E = 200\text{Gpa} = 2 \times 10^5 \text{ Mpa}$$

$$P = 1\text{kN}$$

$$\rho = 7800 \text{ kg/m}^3 = 7800 \times 10^{-9} \text{ kg/mm}^3$$

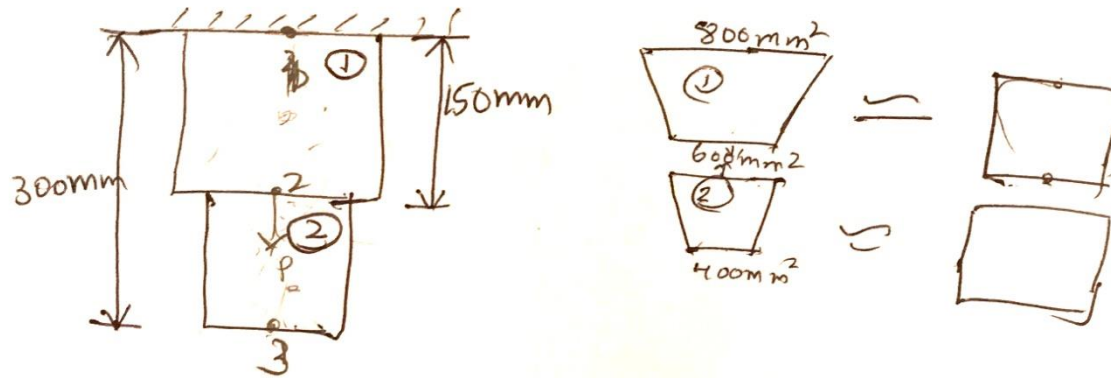
$$t = 10\text{mm}$$

$$A_1 = b \cdot t = 80 \cdot 10 = 800 \text{ mm}^2$$

$$A_3 = b \cdot t = 40 \cdot 10 = 400 \text{ mm}^2$$

$$A_2 = \frac{A_1 + A_3}{2} = \frac{800 + 400}{2} = 600 \text{ mm}^2$$

Conversion of Tapered into stepped bar



Area of element 1

$$A_{\text{element 1}} = \frac{A_1 + A_2}{2} = \frac{800 + 600}{2} = 700 \text{ mm}^2$$

$$A_{\text{element 2}} = \frac{A_3 + A_2}{2} = \frac{400 + 600}{2} = 500 \text{ mm}^2$$

$$F = F_{\text{self weight}} + F_{\text{point load}}$$

$$\text{Self weight of element 1 } F_{\text{self1}} = \rho_1 * A_{\text{ele1}} * l_1 * g = 7800 * 10^{-9} * 700 * 150 * 9.81 = 8.03 \text{ N}$$

$$\text{Self weight of element 2 } F_{\text{self2}} = \rho_2 * A_{\text{ele2}} * l_2 * g = 7800 * 10^{-9} * 500 * 150 * 9.81 = 5.73 \text{ N}$$

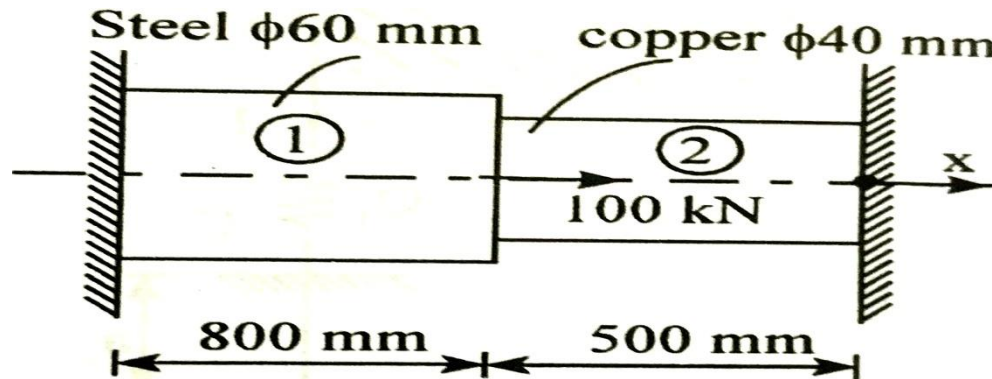
$$\text{Load @ node 1} = F_{\text{self weight 1}} / 2 = 8.03 / 2 = \mathbf{4.015 \text{ N}}$$

$$\begin{aligned} \text{Load @ node 2} &= F_{\text{self weight 1}} / 2 + F_{\text{self weight 2}} / 2 + F_{\text{point load}} \\ &= 8.03 / 2 + 5.73 / 2 + 1 * 10^3 = \mathbf{1006.88 \text{ N}} \end{aligned}$$

$$\text{Load @ node 3} = F_{\text{self weight 2}} / 2 = 5.73 / 2 = \mathbf{2.865 \text{ N}}$$

Penalty approach:

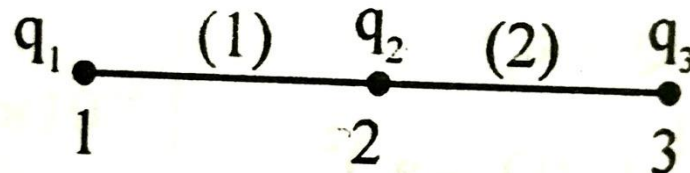
7. A compound bar is loaded as shown in fig, $E_{\text{steel}} = 200\text{GPa}$, $E_{\text{cu}} = 100\text{GPa}$, using penalty method of boundary condition, find a. Nodal displacement, b. Stresses in each element, c. Reaction at the support.



$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi * 60^2}{4} = 2827.43\text{mm}^2$$

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi * 40^2}{4} = 1256.63\text{mm}^2$$

1. Finite element model:



2. Element stiffness matrix for each element:

In general, $[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{2827.43 * 2 * 10^5}{800} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^5 \begin{bmatrix} 7.06 & -7.06 \\ -7.06 & 7.06 \end{bmatrix}$$

Element 2:

$$[K_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = \frac{1256.63 * 1 * 10^5}{500} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^5 \begin{bmatrix} 2.513 & -2.513 \\ -2.513 & 2.513 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^5 \begin{bmatrix} 7.06 & -7.06 & 0 \\ -7.06 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} F_1 + Ca_1 \\ F_2 + Ca_2 \\ F_3 + Ca_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \cdot 10^3 \\ 0 \end{bmatrix}$$

The value C taken as $C = \max[k_{ij}] \cdot 10^4$
 $C = 9.583 \cdot 10^5 \cdot 10^4 = 9.583 \cdot 10^9$

Since there is no displacement so $a_1 = a_2 = a_3 = 0$

5. Equilibrium Equation:

$$[K][q] = [F]$$

$$10^5 \begin{bmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \cdot 10^3 \\ 0 \end{bmatrix}$$

6. Applying Boundary condition:

By Penalty method:

Since the supports are located at node 1 and 3, the co-efficient of stiffness matrix k_{11} & k_{33} are added up with C

Now,

$$10^5 \begin{bmatrix} 9.5837*10^4 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 9.58325*10^4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100*10^3 \\ 0 \end{bmatrix}$$

On solving,

$$q_1 = 7.68*10^{-6} \text{ mm}$$

$$q_2 = 0.104 \text{ mm}$$

$$q_3 = 2.736*10^{-6} \text{ mm}$$

7. Determination of stresses:

Stress: $\sigma = EBq$

$$\sigma_1 = E_1 \frac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \sigma_1 = 2*10^5 * \frac{1}{800} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 7.68*10^{-6} \\ 0.104 \end{bmatrix} \quad \sigma_2 = E_2 \frac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} \quad \sigma_2 = 1*10^5 * \frac{1}{500} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.104 \\ 2.736*10^{-6} \end{bmatrix}$$

$$\sigma_1 = 26.09 \text{ MPa}$$

$$\sigma_2 = -20.8 \text{ MPa}$$

8. Reaction at the support:

$R_n = -C[q_n - a_n]$ (solve for reaction only at the fixed support)

@ node 1: $R_1 = -C[q_1 - a_1]$

$$R_1 = -9.583 \times 10^9 [7.68 \times 10^{-6} - 0]$$

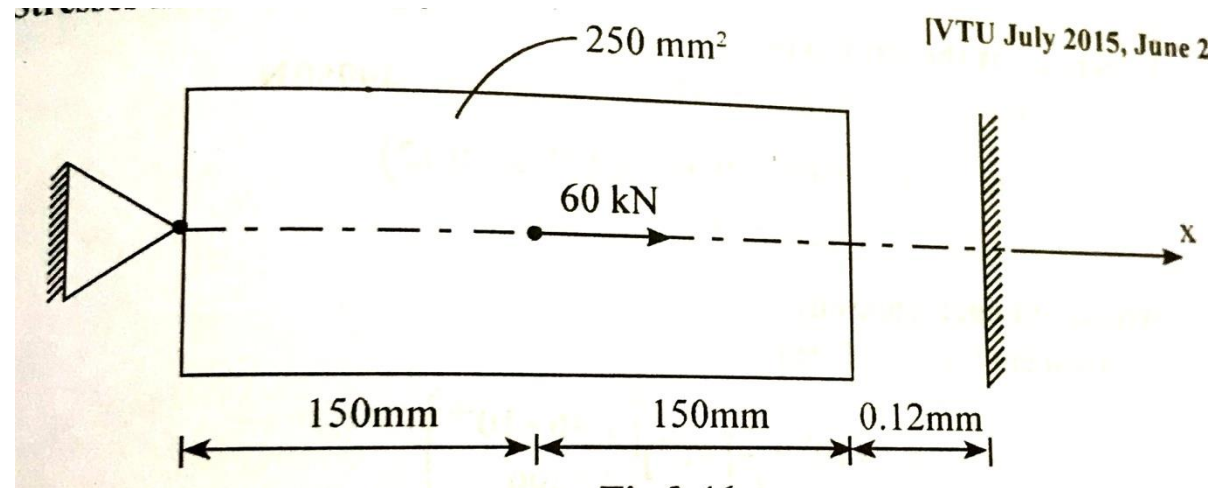
$$R_1 = -73597.44 \text{ N}$$

@ node 3: $R_3 = -C[q_3 - a_3]$

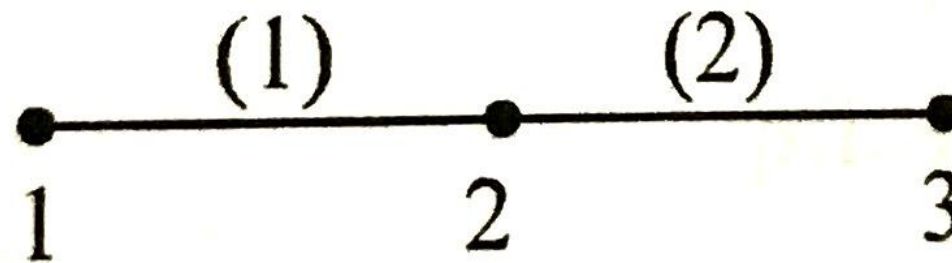
$$R_3 = -9.583 \times 10^9 [2.736 \times 10^{-6} - 0]$$

$$R_3 = -26219.088 \text{ N}$$

8. A bar is loaded as shown in fig $E = 200\text{Gpa}$ by using penalty method find a. Nodal displacement, b. Stresses in each element, c. Reaction at the supports.



1. Finite element model:



2. Element stiffness matrix for each element:

In general, $[K_e] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 1:

$$[K_1] = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = \frac{250 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_1] = 10^5 \begin{bmatrix} 3.33 & -3.33 \\ -3.33 & 3.33 \end{bmatrix}$$

Element 2:

$$[K_2] = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = \frac{250 \times 2 \times 10^5}{150} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [K_2] = 10^5 \begin{bmatrix} 3.33 & -3.33 \\ -3.33 & 3.33 \end{bmatrix}$$

3. Element stiffness matrix into global stiffness matrix:

$$[K_g] = 10^5 \begin{bmatrix} 3.33 & -3.33 & 0 \\ -3.33 & 6.66 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix}$$

4. Displacement vector and Force vector :

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} F_1 + Ca_1 \\ F_2 + Ca_2 \\ F_3 + Ca_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 60 \cdot 10^3 \\ 799.2 \cdot 10^6 \end{bmatrix}$$

The value C taken as $C = \max[k_{ij}] \cdot 10^4$

$$C = 6.66 \cdot 10^5 \cdot 10^4 = 6.66 \cdot 10^9$$

Here there is displacement so $a_3 = 0.12$

5. Equilibrium Equation:

$$[K][q] = [F]$$

$$10^5 \begin{bmatrix} 3.33 & -3.33 & 0 \\ -3.33 & 6.66 & -3.33 \\ 0 & -3.33 & 3.33 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 60 \cdot 10^3 \\ 799.2 \cdot 10^6 \end{bmatrix}$$

6. Applying Boundary condition:

By Penalty method:

Since the supports are located at node 1 and 3, the co-efficient of stiffness matrix k_{11} & k_{33} are added up with C

Now,

$$10^5 \begin{bmatrix} 66603.33 & -3.33 & 0 \\ -7.07 & 6.66 & -3.33 \\ 0 & -3.33 & 66603.33 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 60 \cdot 10^3 \\ 799.2 \cdot 10^6 \end{bmatrix}$$

On solving,

$$q_1 = 7.5 \cdot 10^{-6} \text{ mm}$$

$$q_2 = 0.150095 \text{ mm}$$

$$q_3 = 0.1200015 \text{ mm}$$

7. Determination of stresses:

Stress: $\sigma = EBq$

$$\sigma_1 = E_1 \frac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \sigma_1 = 2 \cdot 10^5 * \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 7.5 \cdot 10^{-6} \\ 0.1499 \end{bmatrix} \quad \sigma_2 = E_2 \frac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} \quad \sigma_2 = 2 \cdot 10^5 * \frac{1}{150} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.1499 \\ 0.120003 \end{bmatrix}$$

$$\sigma_1 = 199 \text{ MPa}$$

$$\sigma_2 = -39.99 \text{ MPa}$$

8. Reaction at the support:

$R_n = -C[q_n - a_n]$ (solve for reaction only at the fixed support)

@ node 1: $R_1 = -C[q_1 - a_1]$

$$R_1 = -6.66 \times 10^9 [7.5 \times 10^{-6} - 0]$$

$$R_1 = -49950 \text{ N}$$

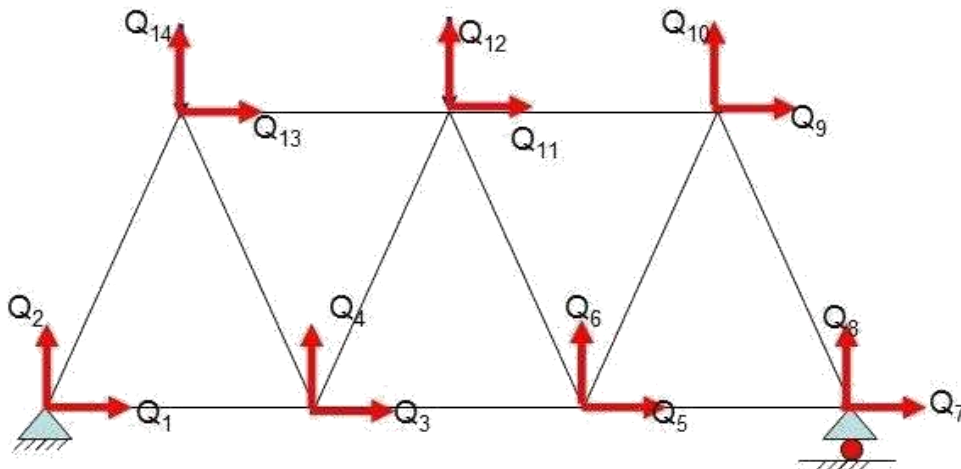
@ node 3: $R_3 = -C[q_3 - a_3]$

$$R_3 = -6.66 \times 10^9 [0.1200015 - 0.12]$$

$$R_3 = -9.99 \times 10^3 \text{ N}$$

Analysis of Truss element:

- A **Truss** is a two force members made up of bars that are connected at the ends by joints.
- Every stress element is in either tension or compression.
- Trusses can be classified as plane truss and space truss.
 - Plane truss is one where the plane of the structure remain in plane even after the application of loads
 - Space truss plane will not be in a same plane

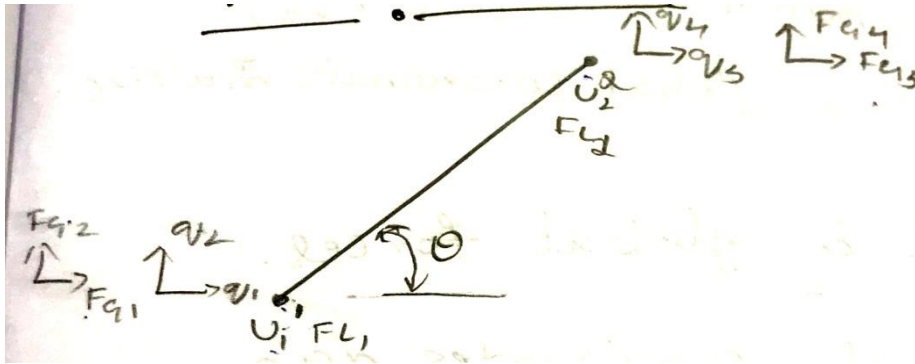


- Fig shows 2d truss structure and each node has two degrees of freedom.
- The only difference between bar element and truss element is that in bars both local and global coordinate systems are same where in truss these are different.

Assumptions made in analysis of trusses:

- Truss element is only a **prismatic member** i.e. cross sectional area is uniform along its length.
- A **load** on a truss can only be applied at the **joints** (nodes).
- Due to the load applied each bar of a truss is either induced with **tensile/compressive forces**.
- The joints in a truss are assumed to be **frictionless pin joints**.
- **Self-weight** of the bars of truss are **neglected** when compared to external load applied.

Derivation of Element stiffness matrix for Truss element (2D):



Let us consider a truss element which is inclined at an angle θ in CCW direction.

The local deflections are U_1 & U_2 .

The global deflections are q_1, q_2, q_3, q_4 .

The relation between the deflections in terms of the trigonometric relation (relation between local and global coordinate)

$$U_1 = q_1 \cos \theta + q_2 \sin \theta \quad \text{----- a}$$

$$U_2 = q_3 \cos \theta + q_4 \sin \theta$$

Replace $\cos\theta = l$ & $\sin\theta = m$,

$$U_1 = q_1 l + q_2 m \text{ ----- b}$$

$$U_2 = q_3 l + q_4 m$$

Equation b in matrix form

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$U_1 = q_1 l + q_2 m + q_3 0 + q_4 0$$

$$U_2 = q_1 0 + q_2 0 + q_3 l + q_4 m$$

$$[U] = [T][q] \text{ ----- 1}$$

Where, $[U]$ is deflection with local coordinate matrix

$[T]$ is the transformation matrix

$[q]$ is deflection with global coordinate matrix

Relationship between local and global forces:

The forces in the local coordinates are F_{L1} & F_{L2}

The forces in the global coordinates are F_{G1} , F_{G2} , F_{G3} , F_{G4}

Work done is independent of the coordinate system

Work done = Force * Displacement

$$[W] = [F_L] * [U]^T = [F_G] * [q]^T \text{ ----- 2}$$

Substitute equation 1 in 2

$$[F_L] * [[T][q]]^T = [F_G] * [q]^T$$

$$[F_L] * [T]^T [q]^T = [F_G] * [q]^T$$

$$[F_L] * [T]^T = [F_G] \text{ ----- 3}$$

We know that the stiffness of local coordinates is given by,

$$[K_L] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

Equilibrium equation

For Local coordinates is $[K_L][U] = [F_L]$ ----- 4

For global coordinates is $[F_L][T]^T = [F_G]$ ----- 5

Substitute equation 4 in 5

$[K_L][U][T]^T = [F_G]$ ----- 6

Substitute equation 1 in 6

$[K_L][T][q][T]^T = [F_G]$

By rearranging,

$[F_G] = [T]^T[K_L][T][q]$

$[F_G] = [K_G][q]$

$$[K_G] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[K_G] = \frac{AE}{L} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[K_G] = \frac{AE}{L} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix}$$

General steps involved in solving a Truss problem:

1. Node data
2. Element connectivity table
3. Element stiffness matrix
4. Displacement Vector and Force vector
5. Global stiffness matrix
6. Equilibrium equation
7. Applying Boundary conditions using elimination approach to find nodal displacement.
8. Calculate stress
9. Finding out reaction at support

Formulae:

1. Length of element: $l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. Element stiffness matrix

$$[K_G] = \frac{AE}{L} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix}$$

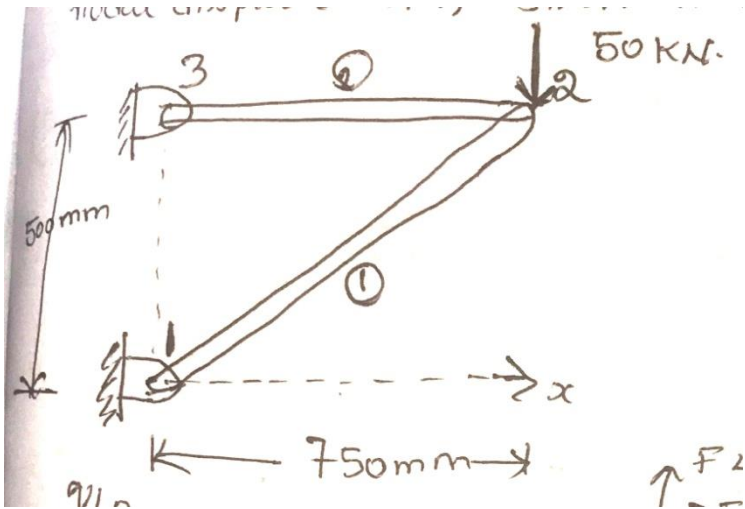
3. $l = \frac{X_{finalnode} - X_{initialnode}}{l_e}$

4. $m = \frac{Y_{finalnode} - Y_{initialnode}}{l_e}$

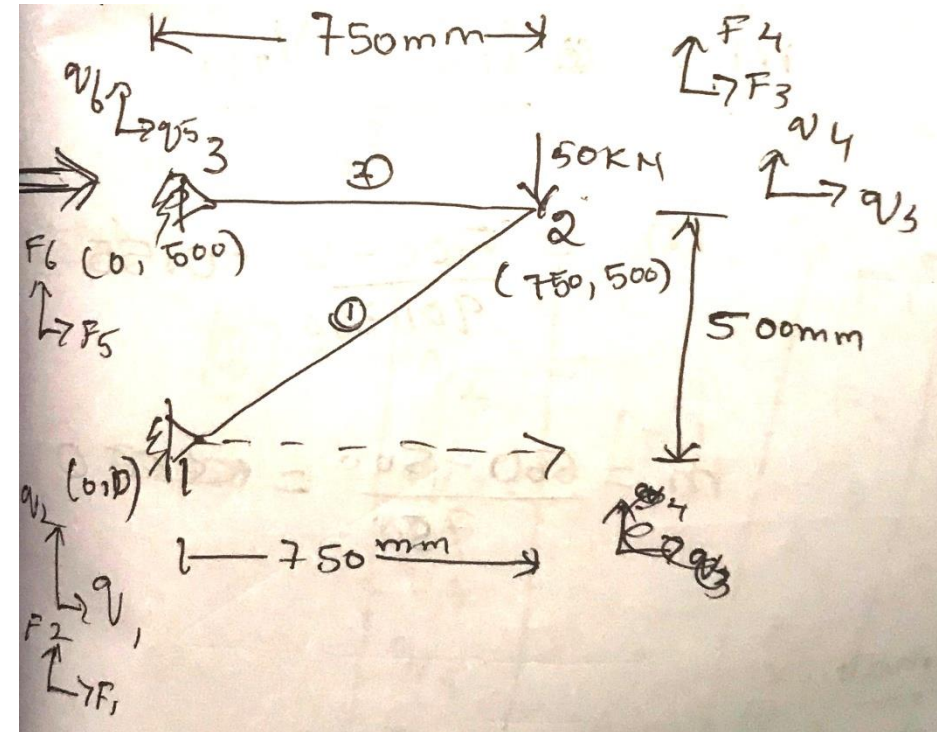
5. Stress: $E B q$ $\sigma_e = E \frac{1}{l_e} [-l \quad -m \quad l \quad m] [q]$

Numerical on Truss element (2D):

1. A truss shown in figure made up of 2 bars. Determine nodal displacement, stress in each element & reaction at the support. $A_1 = 1200\text{mm}^2$, $A_2 = 1000\text{mm}^2$ and $E = E_1 = E_2 = 2 \times 10^5 \text{MPa}$



Line diagram:



1. Node data:

Node no.	X	Y
1	0	0
2	750	500
3	0	500

2. Element connectivity table:

Element no.	Initial Node	Final Node	l_e	l	m
1	1	2	901.38	0.83	0.55
2	2	3	750	-1	0

$$l_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$l_1 = \sqrt{(750 - 0)^2 + (500 - 0)^2} = 901.38 \text{ mm}$$

$$l_2 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$l_2 = \sqrt{(0 - 750)^2 + (500 - 500)^2} = 750 \text{ mm}$$

$$l = \frac{X_{\text{finalnode}} - X_{\text{initialnode}}}{l_e}$$

$$l = \frac{750 - 0}{901.38} = 0.83$$

$$m = \frac{Y_{\text{finalnode}} - Y_{\text{initialnode}}}{l_e}$$

$$m = \frac{500 - 0}{901.38} = 0.55$$

3. Element Stiffness matrix:

$$[K_G] = \frac{AE}{L} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix}$$

For element 1:

$$[K_1] = \frac{1200 \times 2 \times 10^5}{901.38} \begin{bmatrix} 0.83^2 & 0.55 \times 0.83 & -0.83^2 & -0.55 \times 0.83 \\ 0.55 \times 0.83 & 0.55^2 & -0.55 \times 0.83 & -0.55^2 \\ -0.83^2 & -0.55 \times 0.83 & 0.83^2 & 0.55 \times 0.83 \\ -0.55 \times 0.83 & -0.55^2 & 0.55 \times 0.83 & 0.55^2 \end{bmatrix}$$

$$[K_1] = 10^6 \begin{bmatrix} 0.185 & 0.123 & -0.185 & -0.123 \\ 0.123 & 0.081 & -0.123 & -0.55^2 \\ -0.185 & -0.123 & 0.185 & 0.123 \\ -0.123 & -0.081 & 0.123 & 0.081 \end{bmatrix} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix}$$

For element 2:

$$[K_2] = \frac{1000 \cdot 2 \cdot 10^5}{750} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [K_2] = 10^6 \begin{bmatrix} & q_3 & q_4 & q_5 & q_6 \\ 0.27 & 0 & -0.27 & 0 & q_3 \\ 0 & 0 & 0 & 0 & q_4 \\ -0.27 & 0 & 0.27 & 0 & q_5 \\ 0 & -0 & 0 & 0 & q_6 \end{bmatrix}$$

4. Global vector and Force vector:

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50 \cdot 10^3 \\ 0 \\ 0 \end{bmatrix}$$

5. Global stiffness matrix:

$$[K_g] = 10^6 \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ 0.185 & 0.123 & -0.185 & -0.123 & 0 & 0 \\ 0.123 & 0.081 & -0.123 & -0.081 & 0 & 0 \\ -0.185 & -0.123 & 0.455 & 0.123 & -0.27 & 0 \\ -0.123 & -0.081 & 0.123 & 0 & 0 & 0 \\ 0 & 0 & -0.27 & 0 & 0.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix}$$

6. Equilibrium equation:

$$[K][q] = [F]$$

$$10^6 \begin{bmatrix} 0.185 & 0.123 & -0.185 & -0.123 & 0 & 0 \\ 0.123 & 0.081 & -0.123 & -0.081 & 0 & 0 \\ -0.185 & -0.123 & 0.455 & 0.123 & -0.27 & 0 \\ -0.123 & -0.081 & 0.123 & 0 & 0 & 0 \\ 0 & 0 & -0.27 & 0 & 0.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{bmatrix}$$

7. Applying Boundary condition:

Node 1 is fixed, $q_1 = 0$ and $q_2 = 0$

Node 3 is fixed, $q_5 = 0$ and $q_6 = 0$

$$10^6 \begin{bmatrix} 0.185 & 0.123 & -0.185 & -0.123 & 0 & 0 \\ 0.123 & 0.081 & -0.123 & -0.081 & 0 & 0 \\ -0.185 & -0.123 & 0.455 & 0.123 & -0.27 & 0 \\ -0.123 & -0.081 & 0.123 & 0 & 0 & 0 \\ 0 & 0 & -0.27 & 0 & 0.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{bmatrix}$$

$$10^6 \begin{bmatrix} 0.455 & 0.123 \\ 0.123 & 0.081 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -50 \times 10^3 \end{bmatrix}$$

$$q_3 = 0.2830 \text{ mm}$$

$$q_4 = -1.047 \text{ mm}$$

8. Stress: EBq

$$\sigma_1 = E \frac{1}{l_1} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad \sigma_1 = 2 * 10^5 \frac{1}{901.38} \begin{bmatrix} -0.83 & -0.55 & 0.83 & 0.55 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2830 \\ -1.047 \end{bmatrix}$$

$$\sigma_1 = 109.17 \text{MPa}$$

$$\sigma_2 = E \frac{1}{l_2} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad \sigma_2 = 2 * 10^5 \frac{1}{750} \begin{bmatrix} -(-1) & -0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.2830 \\ -1.047 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_2 = -108.16 \text{MPa}$$

9. Reaction at the support:

$$[R] = [K][q] - [F]$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} = 10^6 \begin{bmatrix} 0.185 & 0.123 & -0.185 & -0.123 & 0 & 0 \\ 0.123 & 0.081 & -0.123 & -0.081 & 0 & 0 \\ -0.185 & -0.123 & 0.455 & 0.123 & -0.27 & 0 \\ -0.123 & -0.081 & 0.123 & 0 & 0 & 0 \\ 0 & 0 & -0.27 & 0 & 0.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{bmatrix}$$

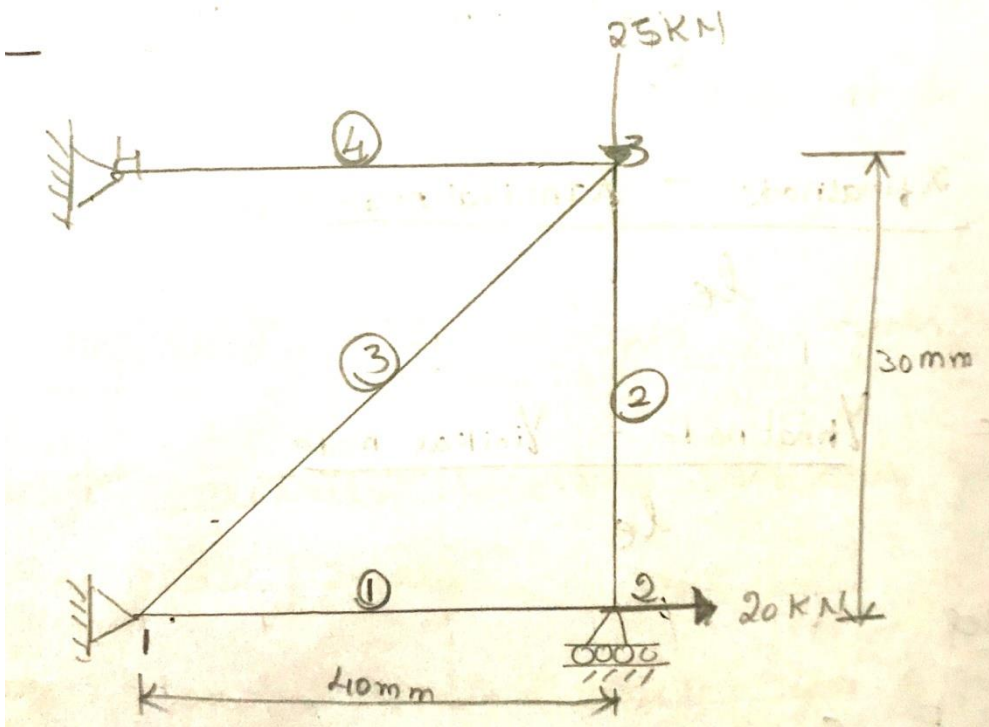
$$R_1 = 73646 \text{ N}$$

$$R_2 = 5000 \text{ N}$$

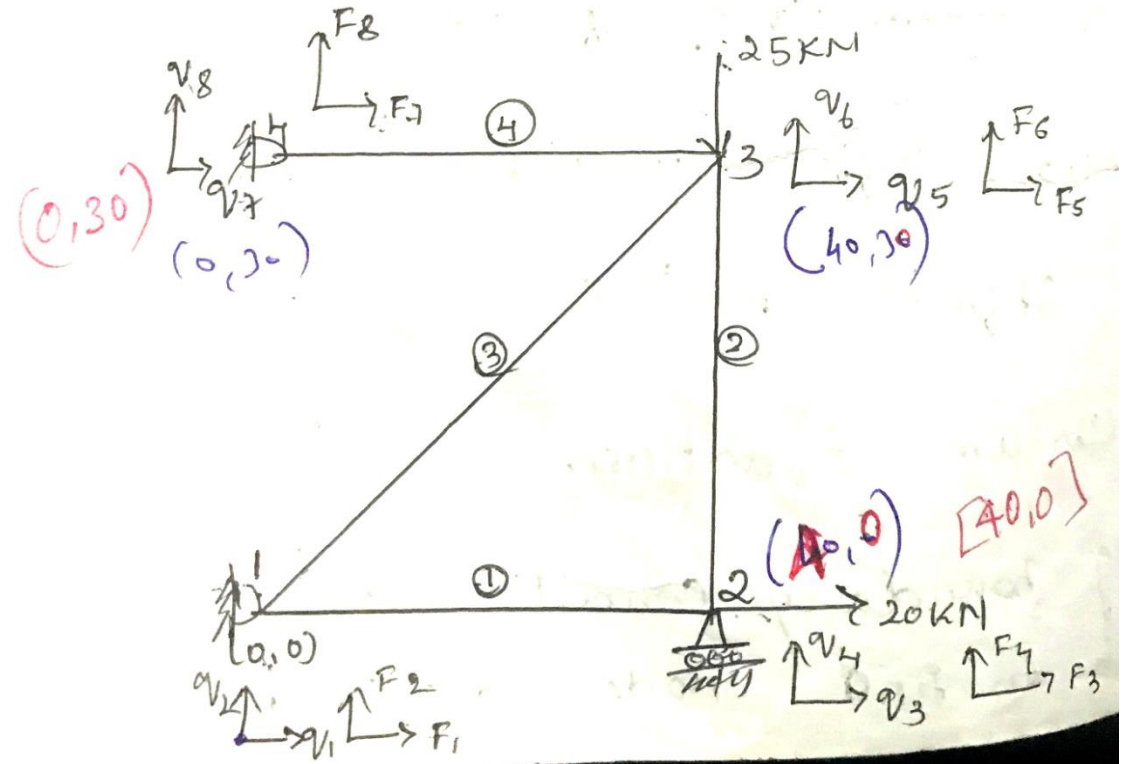
$$R_5 = -73611.9 \text{ N}$$

$$R_3 = R_4 = R_6 = 0$$

2. A 4 bar truss element is shown in fig, Determine the following a. Nodal displacement, b. Stress in each element, c. Reaction at the support. Area of each element = 100mm^2 , Youngs modulus $E = 2 \times 10^5 \text{Mpa}$



Line diagram:



1. Node data:

Node no.	X	Y
1	0	0
2	40	0
3	40	30
4	0	30

2. Element connectivity table:

Element no.	Initial Node	Final Node	I_e	I	m
1	1	2	40	1	0
2	2	3	30	0	1
3	3	1	50	-0.8	-0.6
4	3	4	40	-1	0

To calculate l_e , l , m :

$$l_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$l_1 = \sqrt{(40 - 0)^2 + (0 - 0)^2} = 40mm$$

$$l_2 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$l_2 = \sqrt{(40 - 40)^2 + (30 - 0)^2} = 30mm$$

$$l_3 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$l_3 = \sqrt{(40 - 0)^2 + (30 - 0)^2} = 50mm$$

$$l_4 = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$$

$$l_4 = \sqrt{(0 - 40)^2 + (30 - 30)^2} = 40mm$$

$$l = \frac{X_{finalnode} - X_{initialnode}}{l_e} \quad m = \frac{Y_{finalnode} - Y_{initialnode}}{l_e}$$

$$l = \frac{40 - 0}{40} = 1$$

$$m = \frac{0 - 0}{40} = 0$$

$$l = \frac{40 - 40}{30} = 0$$

$$m = \frac{30 - 0}{30} = 1$$

$$l = \frac{0 - 40}{50} = -0.8$$

$$m = \frac{0 - 30}{50} = -0.6$$

$$l = \frac{0 - 40}{40} = -1$$

$$m = \frac{30 - 30}{40} = 0$$

3. Element Stiffness matrix:

$$[K_G] = \frac{AE}{L} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix}$$

For element 1:

$$[K_1] = \frac{100 \times 2 \times 10^5}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_1] = 10^6 \begin{bmatrix} 0.5 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix}$$

For element 2:

$$[K_2] = \frac{100 \times 2 \times 10^5}{30} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K_1] = 10^6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & -0.6 \\ 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0.6 \end{bmatrix} \begin{matrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix}$$

For Element 3:

$$[K_3] = \frac{100 \times 2 \times 10^5}{50} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$[K_3] = 10^6 \begin{bmatrix} 0.256 & 0.192 & -0.256 & -0.192 \\ 0.192 & 0.144 & -0.192 & -0.144 \\ -0.256 & -0.192 & 0.256 & 0.192 \\ -0.192 & -0.144 & 0.192 & 0.144 \end{bmatrix} \begin{matrix} q_5 \\ q_6 \\ q_1 \\ q_2 \end{matrix}$$

For element 4:

$$[K_4] = \frac{100 \times 2 \times 10^5}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_4] = 10^6 \begin{bmatrix} 0.5 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{matrix}$$

4. Global Vector and Force Vector:

$$[q] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} \quad [F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \times 10^3 \\ 0 \\ 0 \\ -25 \times 10^3 \\ 0 \\ 0 \end{bmatrix}$$

5. Global stiffness matrix

$$[K] = 10^6 \begin{bmatrix} 0.756 & 0.192 & -0.5 & 0 & -0.256 & -0.192 & 0 & 0 \\ 0.192 & 0.144 & 0 & 0 & -0.192 & -0.144 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & -0.6 & 0 & 0 \\ -0.256 & -0.192 & 0 & 0 & 0.756 & -0.192 & 0.5 & 0 \\ -0.192 & -0.144 & 0 & -0.6 & -0.192 & 0.744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. Equilibrium equation:

$$[K][q]=[F]$$

$$10^6 \begin{bmatrix} 0.756 & 0.192 & -0.5 & 0 & -0.256 & -0.192 & 0 & 0 \\ 0.192 & 0.144 & 0 & 0 & -0.192 & -0.144 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & -0.6 & 0 & 0 \\ -0.256 & -0.192 & 0 & 0 & 0.756 & -0.192 & 0.5 & 0 \\ -0.192 & -0.144 & 0 & -0.6 & -0.192 & 0.744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{bmatrix}$$

7. Applying Boundary condition:

Node 1 is fixed, $q_1 = 0$ and $q_2 = 0$

Node 2 is fixed, $q_4 = 0$

Node 4 is fixed, $q_7 = 0$ and $q_8 = 0$

$$10^6 \begin{bmatrix} 0.756 & 0.192 & -0.5 & 0 & -0.256 & -0.192 & 0 & 0 \\ 0.192 & 0.144 & 0 & 0 & -0.192 & -0.144 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & -0.6 & 0 & 0 \\ -0.256 & -0.192 & 0 & 0 & 0.756 & -0.192 & 0.5 & 0 \\ -0.192 & -0.144 & 0 & -0.6 & -0.192 & 0.744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{bmatrix}$$

$$10^6 \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.756 & 0.192 \\ 0 & 0.192 & 0.744 \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 20 \cdot 10^3 \\ 0 \\ -25 \cdot 10^3 \end{bmatrix}$$

$$q_3 = 0.04 \text{ mm}$$

$$q_5 = 9.132 \cdot 10^{-3} \text{ mm}$$

$$q_6 = -0.0359 \text{ mm}$$

8. Stress:

For element 1:

$$\sigma_1 = E \frac{1}{l_1} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$\sigma_1 = 2 \cdot 10^5 \frac{1}{40} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.04 \\ 0 \end{bmatrix}$$

$$\sigma_1 = 200 \text{ Mpa}$$

For Element 2:

$$\sigma_2 = E \frac{1}{l_2} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\sigma_2 = 2 * 10^5 \frac{1}{30} \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0 \\ 9.132 * 10^{-3} \\ -0.0359 \end{bmatrix}$$

$$\sigma_2 = -239.33 \text{ MPa}$$

For Element 3:

$$\sigma_3 = E \frac{1}{l_3} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_5 \\ q_6 \\ q_1 \\ q_2 \end{bmatrix}$$

$$\sigma_3 = 2 * 10^5 \frac{1}{50} \begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 9.132 * 10^{-3} \\ -0.0359 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_3 = -56.93 \text{ MPa}$$

For Element 4:

$$\sigma_4 = E \frac{1}{l_4} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix}$$

$$\sigma_4 = 2 * 10^5 \frac{1}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 9.132 * 10^{-3} \\ -0.0359 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_4 = 45.66 \text{ MPa}$$

9. Reaction at the support:

$$[R]=[K][q]-[F]$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} 10^6 \begin{bmatrix} 0.756 & 0.192 & -0.5 & 0 & -0.256 & -0.192 & 0 & 0 \\ 0.192 & 0.144 & 0 & 0 & -0.192 & -0.144 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & -0.6 & 0 & 0 \\ -0.256 & -0.192 & 0 & 0 & 0.756 & -0.192 & 0.5 & 0 \\ -0.192 & -0.144 & 0 & -0.6 & -0.192 & 0.744 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 20*10^3 \\ 0 \\ 0 \\ -25*10^3 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 = -15810\text{N}$$

$$R_2 = 3130 \text{ N}$$

$$R_4 = 21800\text{N}$$

$$R_7 = -4110 \text{ N}$$

$$R_3 = R_5 = R_6 = R_8 = 0$$

Shape Function:

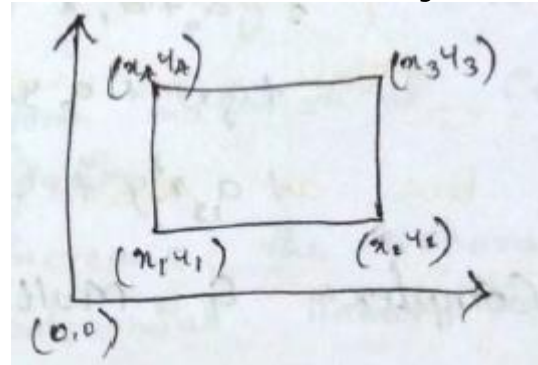
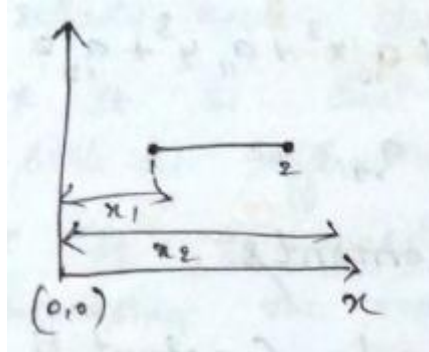
Shape function relates the field variable at any point within the element to the field variables of nodal points. This is also called as interpolation function and approximating function. These are denoted by “**N**”.

Properties of Shape function:

- Shape function has a value of unity at its own node and Zero at other node.
- The sum of shape function is unity.
- The derivative of shape function is a constant.
- Shape function is linearly independent.
- Shape functions are generally represented in natural coordinates system.
- The number of shape function for a given element will be the same as the number of nodes present in an element.

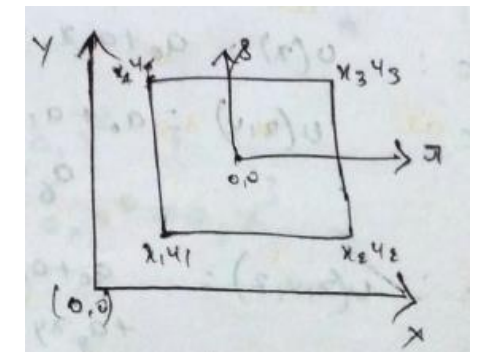
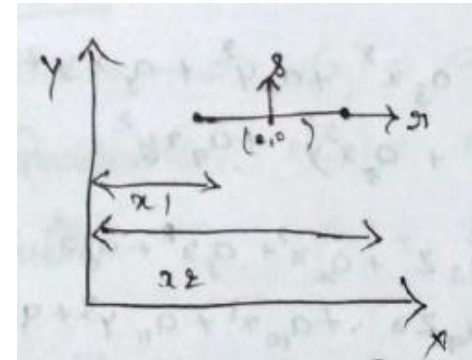
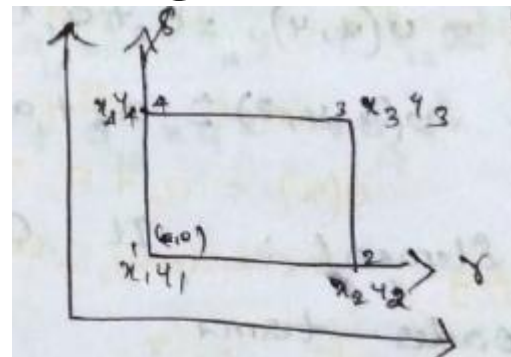
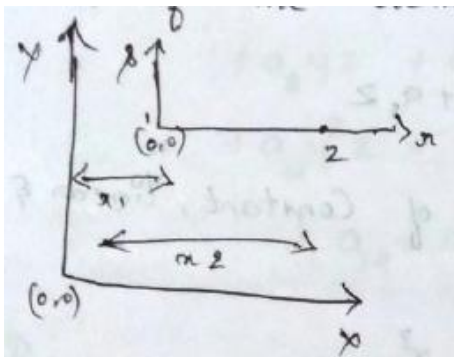
Global coordinate system: It is used to define points and nodes of entire body.

Cartesian – Polar or Spherical coordinate system



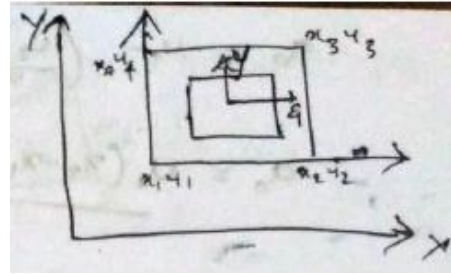
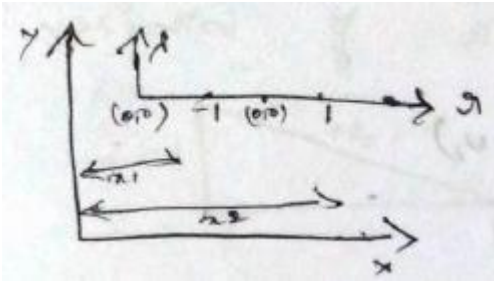
Local coordinate system:

The origin of these system is located either at node of the element or at the centre of the element as shown in figure



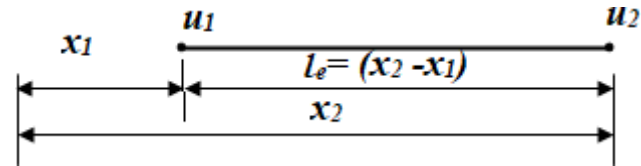
Natural Coordinate System:

A local coordinate system that permits specification of a point within the element by dimensionless or normalized number whose absolute magnitude never exceeds unity.



Derive shape function for a 1D-bar element.

A. Global Coordinate system:



Consider a 1D bar element having two nodes 1 and 2, at each node there is one DOF.

Let N_1 and N_2 be the shape function at node 1 and node 2 respectively.

Let u_1 and u_2 be the displacement at node 1 and node 2 respectively.

Let x_1 and x_2 be the distances from reference to node 1 and node 2 respectively.

Consider a displacement function for 1D bar element,

$$u = a_0 + a_1 x \text{ ----- (A)}$$

At node 1:- $u = u_1$ and $x = x_1$

$$u_1 = a_0 + a_1 x_1 \text{ ----- (1)}$$

At node 2:- $u = u_2$ and $x = x_2$

$$u_2 = a_0 + a_1 x_2 \text{ ----- (2)}$$

By Subtracting equations (2) in (1)

$$a_1 = \frac{u_2 - u_1}{x_2 - x_1}$$

Substituting a_1 in equation 1

$$a_0 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}$$

Substituting values of a_0 and a_1 in equation (A)

$$u = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} + \left(\frac{u_2 - u_1}{x_2 - x_1} \right) x$$

$$u_2 = a_0 + a_1 x_2$$

$$u_1 = a_0 + a_1 x_1$$

$$u_2 - u_1 = a_1 (x_2 - x_1)$$

$$a_1 = \frac{u_2 - u_1}{x_2 - x_1}$$

On solving:

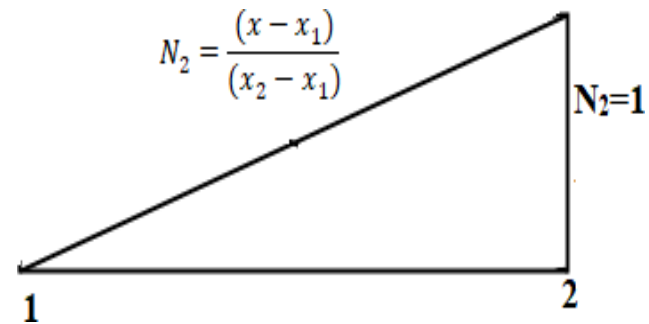
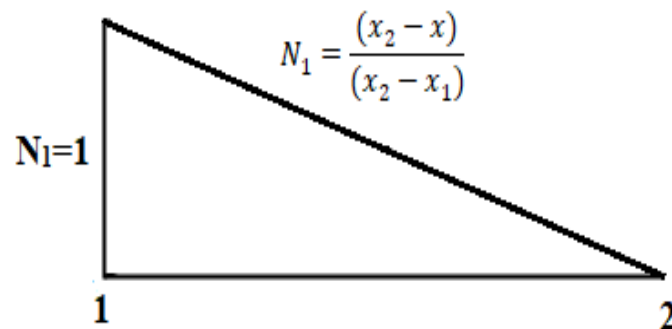
$$u = \frac{u_1 x_2 - u_2 x_1 + u_2 x - u_1 x}{x_2 - x_1}$$

$$u = u_1 \frac{(x_2 - x)}{x_2 - x_1} + u_2 \frac{(x - x_1)}{x_2 - x_1}$$

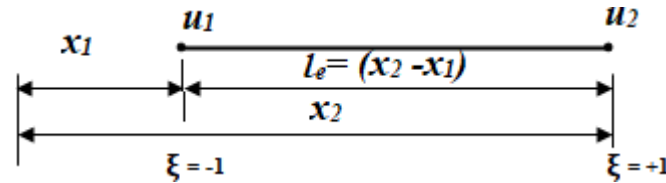
$$u = u_1 N_1 + u_2 N_2$$

Where, $N_1 = \frac{(x_2 - x)}{x_2 - x_1}$ $N_2 = \frac{(x - x_1)}{x_2 - x_1}$

These are the shape functions of Node 1 and Node 2 respectively.



B. Natural coordinate system:



Consider a 1D-bar element as shown in figure having two nodes and at each node there is one DOF.

Let N_1 and N_2 be the shape function at node 1 and node 2 respectively.

Let u_1 and u_2 be the displacement at node 1 and node 2 respectively.

Let the value of $\xi = -1$ and $\xi = +1$ at node 1 and node 2 respectively.

$$u = a_0 + a_1 \xi \text{ ----- (A)}$$

At node 1:- $u = u_1$ and $\xi = -1$

$$u_1 = a_0 - a_1 \text{ ----- (1)}$$

At node 2:- $u = u_2$ and $\xi = 1$

$$u_2 = a_0 + a_1 \text{ ----- (2)}$$

By Subtracting equations (2) in (1)

$$a_0 = \frac{u_1 + u_2}{2} \quad a_1 = \frac{u_2 - u_1}{2}$$

Substituting values of a_0 and a_1 in equation (A)

$$u = \frac{u_1 + u_2}{2} + \left(\frac{u_2 - u_1}{2} \right) \xi$$

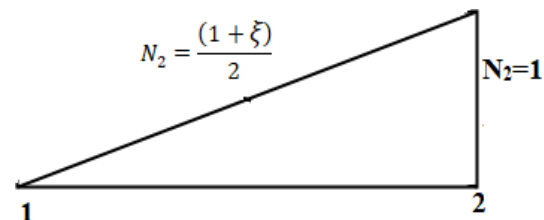
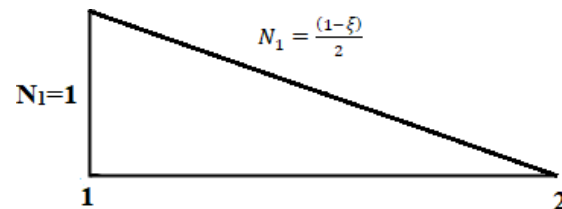
On solving,

$$u = \frac{(1-\xi)}{2} u_1 + \frac{(1+\xi)}{2} u_2$$

$$u = N_1 u_1 + N_2 u_2$$

Where, $N_1 = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2}$

These are the shape functions of Node 1 and Node 2 respectively



Derive shape function for a 2D - Triangular element

A. Global coordinate System:

Consider 2D element as shown figure

Assuming trial displacement function:

$$u(x,y) = a_0 + a_1x + a_2y$$

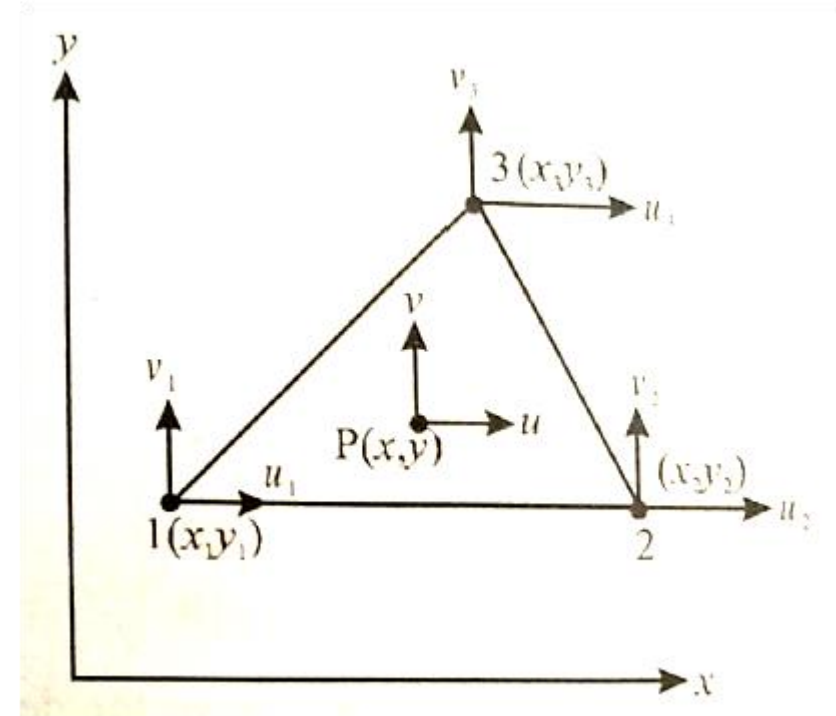
In matrix form,

$$u = [1 \quad x \quad y] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad \text{---- 1}$$

Applying Boundary condition:

@ node 1: $x = x_1, u = u_1, y = y_1$

$$u_1 = a_0 + a_1x_1 + a_2y_1 \quad \text{----- 2}$$



@ node 2: $x = x_2, u = u_2, y = y_2$

$$u_2 = a_0 + a_1 x_2 + a_2 y_2 \text{ ----- 3}$$

@ node 3: $x = x_3, u = u_3, y = y_3$

$$u_3 = a_0 + a_1 x_3 + a_2 y_3 \text{ ----- 4}$$

Writing equation 2,3,4 in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$A^{-1} = \text{adj}(a) / [A]$$

In matrix Area of triangle = $\frac{1}{2} [A]$

$$[A] = 2A_e$$

$$\text{adj}(A) = C^T$$

$$C = (-1)^{i+j} m_{ij}$$

$$C_{11} = (-1)^{1+1} (x_2 y_3 - x_3 y_2)$$

$$C_{12} = (-1)^{1+2} (y_3 - y_2)$$

$$C_{13} = (-1)^{1+3} (x_3 - x_2)$$

$$C_{21} = (-1)^{2+1} (x_1 y_3 - x_3 y_1)$$

$$C_{22} = (-1)^{2+2} (y_3 - y_1)$$

$$C_{23} = (-1)^{2+3} (x_3 - x_1)$$

$$C_{31} = (-1)^{3+1} (x_1 y_2 - x_2 y_1)$$

$$C_{32} = (-1)^{3+2} (y_2 - y_1)$$

$$C_{33} = (-1)^{3+3} (x_2 - x_1)$$

By arranging in matrix form:

$$[C] = \begin{bmatrix} (\alpha_2 y_3 - \alpha_3 y_2) & (y_3 - y_2) & (\alpha_3 - \alpha_2) \\ (\alpha_3 y_1 - \alpha_1 y_3) & (y_3 - y_1) & (\alpha_1 - \alpha_3) \\ (\alpha_1 y_2 - \alpha_2 y_1) & (y_1 - y_2) & (\alpha_2 - \alpha_1) \end{bmatrix}$$

$$[C] = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \quad [C]^T = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \quad A^{-1} = \frac{1}{2A_e} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \frac{1}{2A_e} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Substitute a_0, a_1, a_2 in equation 1

Therefore,

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A_e} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \frac{1}{2A_e} \begin{bmatrix} (\alpha_1 + x\beta_1 + y\gamma_1) & (\alpha_2 + x\beta_2 + y\gamma_2) & (\alpha_3 + x\beta_3 + y\gamma_3) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

where

$$N_1 = \frac{1}{2A_e} (\alpha_1 + x\beta_1 + y\gamma_1)$$

$$N_2 = \frac{1}{2A_e} (\alpha_2 + x\beta_2 + y\gamma_2)$$

$$N_3 = \frac{1}{2A_e} (\alpha_3 + x\beta_3 + y\gamma_3)$$

Derive shape function for a 2D - Triangular element

B. Natural coordinate System:

Consider 2D element as shown figure

Assuming trial displacement function:

$$u(x,y) = a_0 + a_1\xi + a_2\eta$$

In matrix form, $u = [1 \quad \xi \quad \eta] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ ----- 1

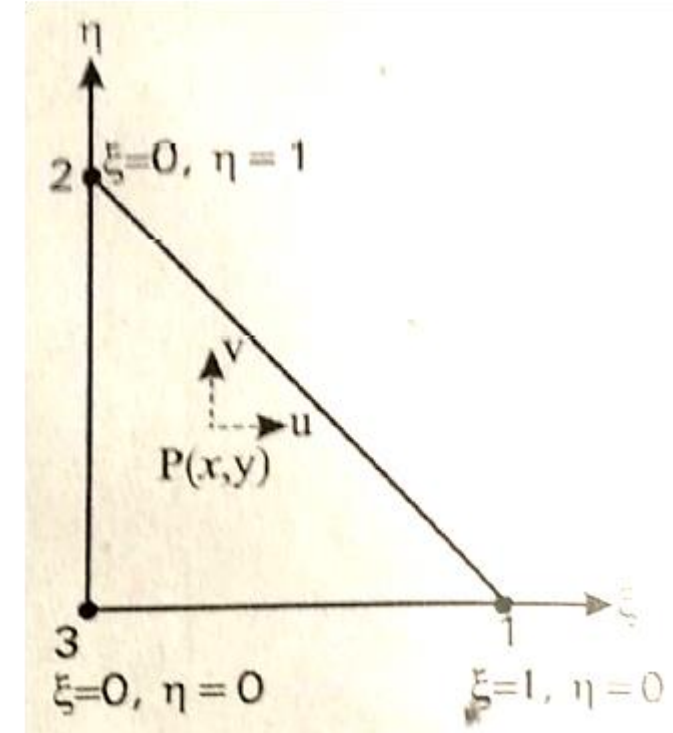
Applying Boundary condition:

@ node 1: $u = u_1, \xi = 1, \eta = 0$

$$u_1 = a_0 + a_1 \text{ ----- 2}$$

@ node 2: $u = u_2, \xi = 0, \eta = 1$

$$u_2 = a_0 + a_2 \text{ ----- 3}$$



@ node 2: $u = u_2$, $\xi = 0$, $\eta = 0$

$$u_3 = a_0 \text{ ----- } 4$$

Writing equation 2,3,4 in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$A^{-1} = \text{adj}(a)/[A]$$

$$[A] = 1$$

$$\text{adj}(A) = C^T$$

$$C = (-1)^{i+j} m_{ij}$$

$$C_{11} = (-1)^{1+1} (0-0) = 0$$

$$C_{12} = (-1)^{1+2} (0-1) = +1$$

$$C_{13} = (-1)^{1+3} (0-0) = 0$$

$$C_{21} = (-1)^{2+1} (0-0) = 0$$

$$C_{22} = (-1)^{2+2} (0-0) = 0$$

$$C_{23} = (-1)^{2+3} (0-1) = +1$$

$$C_{31} = (-1)^{3+1} (1-0) = 1$$

$$C_{32} = (-1)^{3+2} (1-0) = -1$$

$$C_{33} = (-1)^{3+3} (0-1) = -1$$

By rearranging in matrix form:

$$[C] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad [C]^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Substitute a_0, a_1, a_2 in equation 1

$$u = \begin{bmatrix} 1 & \xi & \eta \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

On solving,

$$u = \begin{bmatrix} \xi & \eta & 1-\xi-\eta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u = \xi u_1 + \eta u_2 + (1-\xi-\eta)u_3$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

therefore,

$$N_1 = \xi, N_2 = \eta \text{ and } N_3 = 1-\xi-\eta$$

are the shape functions at node 1, node 2 and node 3 respectively

Numerical on shape function:

1. A nodal coordinate of a triangular element as shown in fig, the x – coordinate of interior point P is 3.3 and the shape function $N_1 = 0.3$. Determine N_2 , N_3 & y coordinate.

Given,

$$N_1 = 0.3, x = 3.3, x_1 = 1, x_2 = 5, x_3 = 4, y_1 = 2, y_2 = 3, y_3 = 6 \quad N_2 = ?, N_3 = ?$$

Considering geometric shape function

$$x = N_1x_1 + N_2x_2 + N_3x_3$$

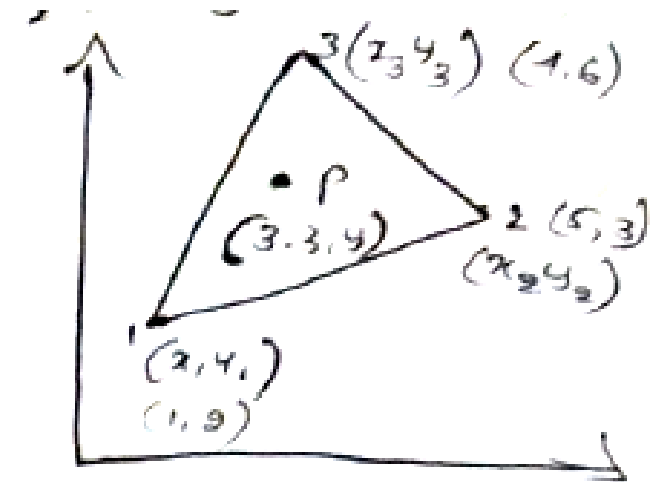
$$3.3 = 0.3(1) + N_2(5) + N_3(4)$$

$$\text{Wkt, } N_1 = \xi = 0.3, N_2 = \eta \text{ and } N_3 = 1 - \xi - \eta$$

Therefore,

$$3.3 = 0.3(1) + \eta(5) + (1 - 0.3 - \eta)(4)$$

$$\eta = 0.2 = N_2$$



$$N_3 = 1 - \xi - \eta$$

$$N_3 = 1 - 0.3 - 0.2$$

$$N_3 = \mathbf{0.5}$$

Now consider,

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

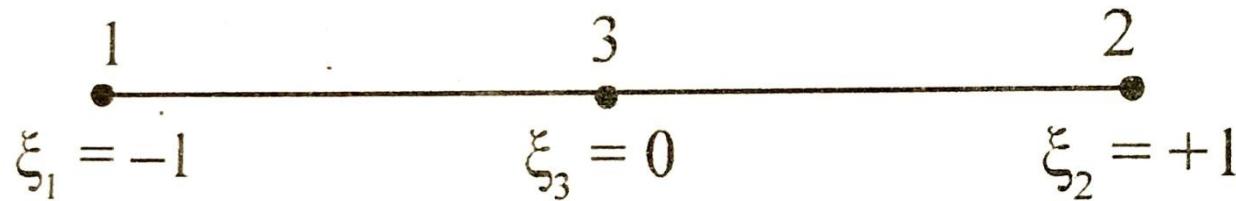
$$y = 0.3 * 2 + 0.2 * 3 + (1 - 0.3 - 0.2) * 6$$

$$y = \mathbf{4.2}$$

Higher order interpolation function:

a. Quadratic bar element (3 noded axial element):

Consider 1-D bar element shown in figure, it has 3 nodes and each nodes is subjected to 1 DOF



The shape function is derived as below:

@ node 1:

$$N_1 = L_1(\xi) = \frac{(\xi - \xi_3)(\xi - \xi_2)}{(\xi_1 - \xi_3)(\xi_1 - \xi_2)} \quad N_1 = L_1(\xi) = \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} \quad N_1 = L_1(\xi) = \frac{\xi(\xi - 1)}{2}$$

@ node 2:

$$N_2 = L_2(\xi) = \frac{(\xi - \xi_3)(\xi - \xi_1)}{(\xi_2 - \xi_3)(\xi_2 - \xi_1)}$$

$$N_2 = L_2(\xi) = \frac{(\xi - 0)(\xi + 1)}{(1 - 0)(1 + 1)}$$

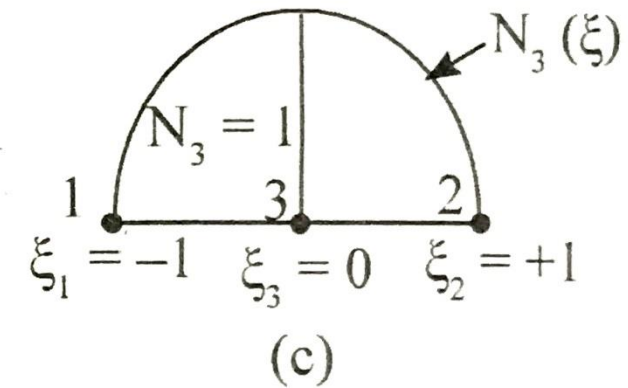
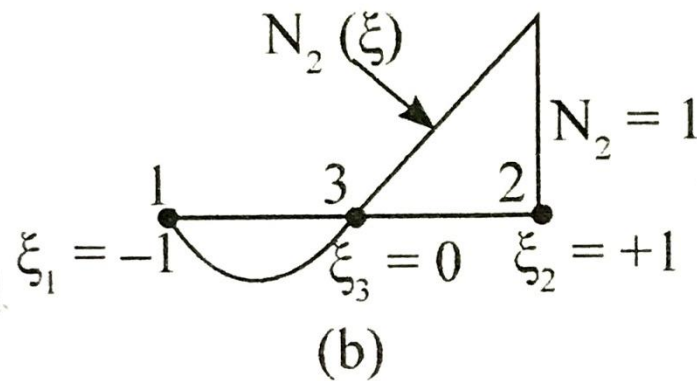
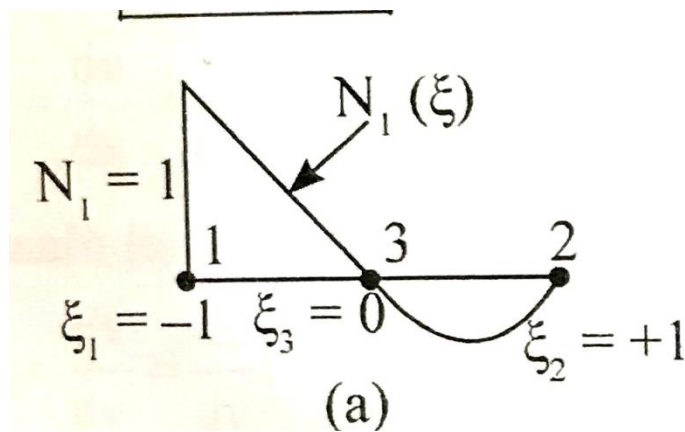
$$N_2 = L_2(\xi) = \frac{\xi(\xi + 1)}{2}$$

@ node 3:

$$N_3 = L_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)}$$

$$N_3 = L_3(\xi) = \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)}$$

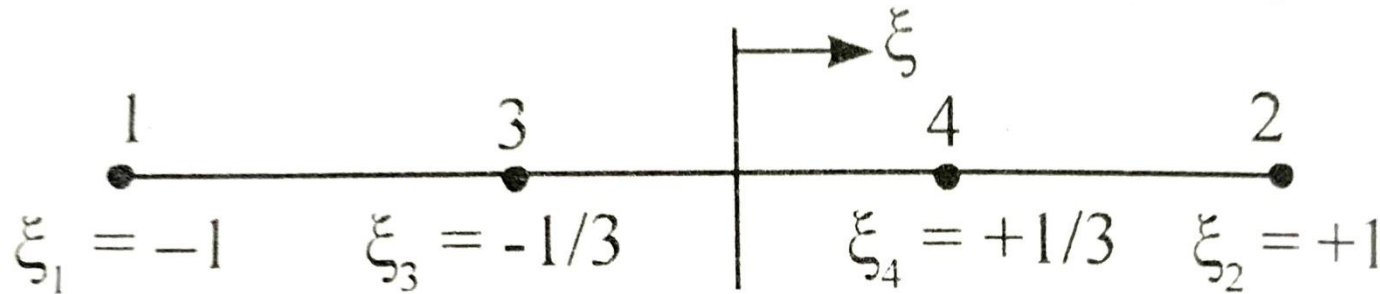
$$N_3 = L_3(\xi) = \frac{\xi^2 - 1}{-1} = 1 - \xi^2$$



Higher order interpolation function:

b. Cubic bar element :

Consider a cubic 1-D bar element shown in figure, it has 4 nodes and each nodes is subjected to 1 DOF



The shape function of nodes 1,2,3,4 is derived as follows:

@ node 1:

$$N_1 = \frac{(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_2)}{(\xi_1 - \xi_3)(\xi_1 - \xi_4)(\xi_1 - \xi_2)}$$

$$N_1 = \frac{(\xi + 1/3)(\xi - 1/3)(\xi - 1)}{(-1 + 1/3)(-1 - 1/3)(-1 - 1)}$$

$$N_1 = -\frac{9}{16}(\xi^2 - 1/9)(\xi - 1)$$

@ node 2:

$$N_2 = \frac{(\xi - \xi_4)(\xi - \xi_3)(\xi - \xi_1)}{(\xi_2 - \xi_4)(\xi_2 - \xi_3)(\xi_2 - \xi_1)}$$

$$N_2 = \frac{(\xi - 1/3)(\xi + 1/3)(\xi + 1)}{(1 - 1/3)(1 + 1/3)(1 + 1)}$$

$$N_2 = \frac{9}{16}(\xi^2 - 1/9)(\xi + 1)$$

@ node 3:

$$N_3 = \frac{(\xi - \xi_1)(\xi - \xi_4)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_4)(\xi_3 - \xi_2)}$$

$$N_3 = \frac{(\xi + 1)(\xi - 1/3)(\xi - 1)}{(-1/3 + 1/3)(-1/3 - 1/3)(-1/3 - 1)}$$

$$N_3 = \frac{27}{16}(\xi^2 - 1)(\xi - 1/3)$$

@ node 4:

$$N_4 = \frac{(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_1)}{(\xi_4 - \xi_2)(\xi_4 - \xi_3)(\xi_4 - \xi_1)}$$

$$N_4 = \frac{(\xi - 1)(\xi + 1/3)(\xi + 1)}{(1/3 - 1)(1/3 + 1/3)(1/3 + 1)}$$

$$N_4 = \frac{27}{16}(\xi^2 - 1)(\xi + 1/3)$$

Constant Strain Triangular element (CST- Element):

Fig shows a CST element in cartesian coordinates x and y having 3 – nodes, at each node 2 DOF i.e., x – displacement as ‘u’ and y displacement as ‘v’.

For a triangular element, displacement function is as follows,

$$u(x,y) = a_1 + a_2x + a_3y \text{ -----in x – direction}$$

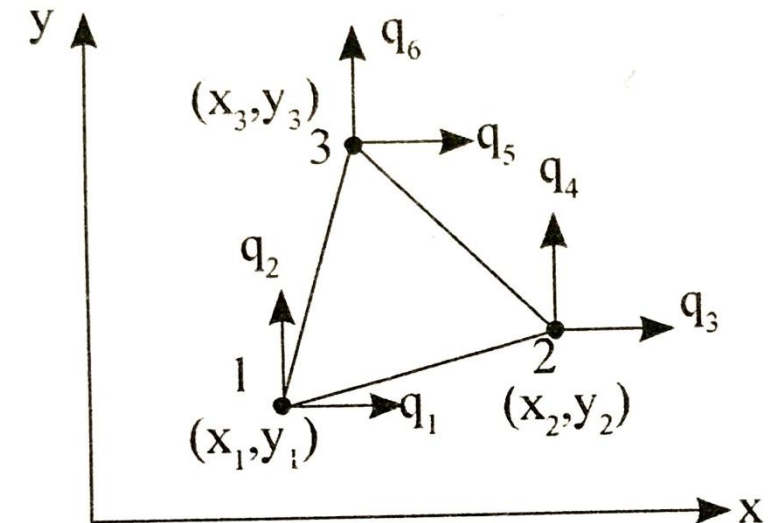
$$v(x,y) = b_1 + b_2x + b_3y \text{ ----- in y - direction}$$

Strain in x – direction:

$$\varepsilon_x = \frac{du}{dx} = \frac{d}{dx}(a_1 + a_2x + a_3y) = a_2$$

Strain in y – direction:

$$\varepsilon_y = \frac{dv}{dy} = \frac{d}{dy}(b_1 + b_2x + b_3y) = b_3$$



Hence strain is constant with triangular element irrespective of its position. As per Hooke's law, $\sigma = E \cdot \varepsilon$, where, E = young's modulus

For any triangle, if σ and E are constant, then automatically the strain ε is also constant in that triangle and hence called **Constant Strain Triangle**.

Application of CST element:

1. Used in areas where the strain is small
2. Used in mesh transition area
3. Avoid using CST element in stress concentration or other crucial areas in the structure, such as edges of holes and corners
4. Recommended for quick and preliminary FE analysis

Linear Strain Triangular element (LST – element)

in case of LST element, displacement function is as follows,

$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \text{ ----- in } x \text{ – direction}$$

$$v(x,y) = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 \text{ ----- in } y \text{ - direction}$$

Strain in x – direction:

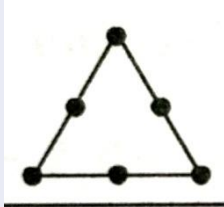
$$\varepsilon_x = \frac{du}{dx} = \frac{d}{dx}(a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2) = a_2 + 2a_4x + a_5y$$

Strain in y – direction

$$\varepsilon_y = \frac{dv}{dy} = \frac{d}{dy}(b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2) = b_3 + b_5x + 2b_6y$$

Hence, strain and corresponding stresses vary linearly with the coordinates. Triangular elements having linear stress/strain variation within elements are called as Linear Strain Triangular Elements

Difference between CST and LST element:

Sl. No	CST	LST
1	It is a 2D linear element (Simplex element)	It is 2D non – linear element (Complex element)
2	It has only three nodes at the corners 	It has only three primary nodes at the corner and three secondary nodes at the mid sides 
3	The strain is constant through out the element	The strain is varying linearly inside the element
4	The displacement function for CST is $u(x,y) = a_1 + a_2x + a_3y$ $v(x,y) = b_1 + b_2x + b_3y$	The displacement function for LST is $u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$ $v(x,y) = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2$

4 Noded Tetrahedral Element (TET – 4):

Consider a tetrahedral element shown in fig with corner nodes 1,2,3 and 4.

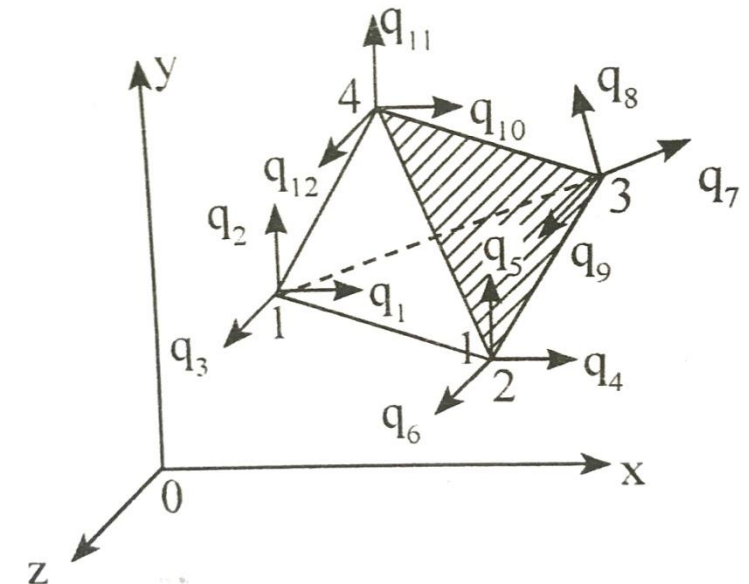
The element is four node solid

The nodes of the element must be numbered such that when viewed from the last node (say node 4), the first three nodes are numbered in a CCW direction

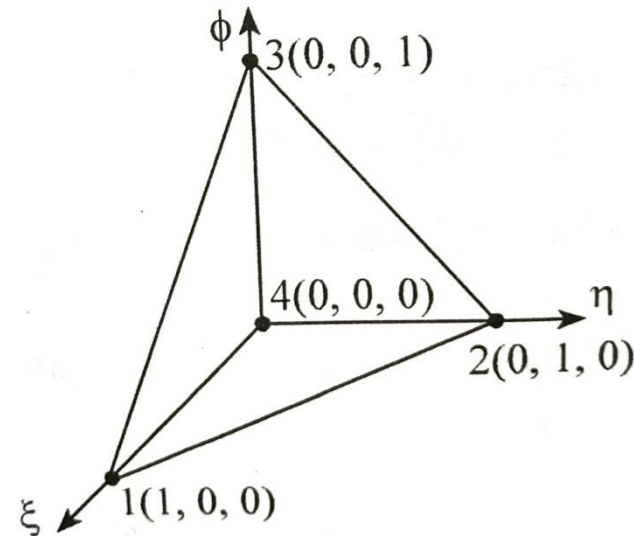
This ordering of nodes avoids the calculation of negative volumes

The nodal displacement vector is given by,

$$q = [q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12}]^T$$



The shape functions for tetrahedral element in natural coordinates ξ , η and ϕ can be determined as follows,



Shape function @ node1:

$$N_1 = \frac{\xi - \xi_4}{\xi_1 - \xi_4} = \frac{\xi - 0}{1 - 0} = \xi$$

Shape function @ node2:

$$N_2 = \frac{\eta - \eta_4}{\eta_2 - \eta_4} = \frac{\eta - 0}{1 - 0} = \eta$$

Shape function @ node3:

$$N_3 = \frac{\phi - \phi_4}{\phi_3 - \phi_4} = \frac{\phi - 0}{1 - 0} = \phi$$

Shape function @ node4:

$$N_4 = \frac{(\xi - \xi_1)(\eta - \eta_2)(\phi - \phi_3)}{(\xi_4 - \xi_1)(\eta_4 - \eta_2)(\phi_4 - \phi_3)}$$

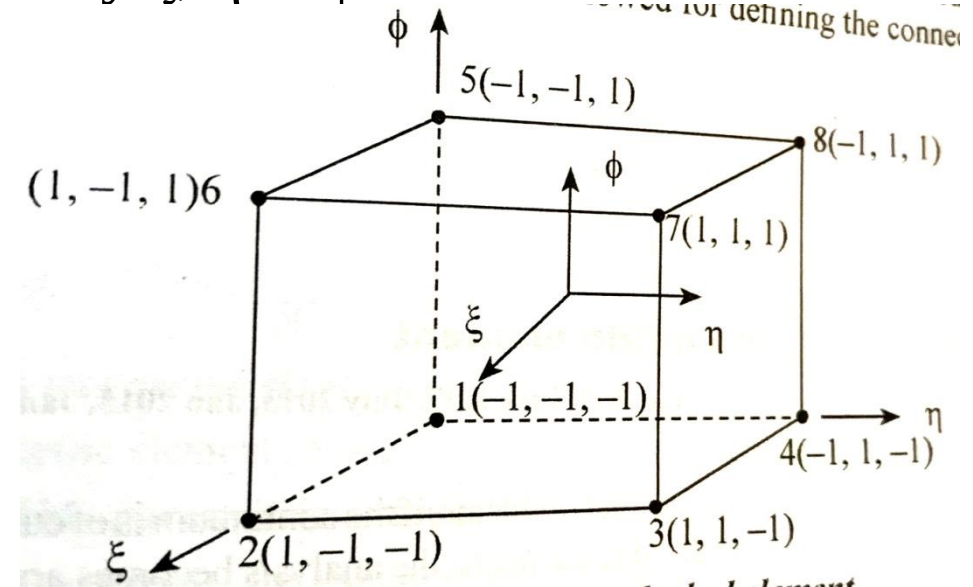
$$N_4 = \frac{(\xi - 1)(\eta - 1)(\phi - 1)}{(-1)(-1)(-1)}$$

$$N_4 = 1 - \eta - \xi - \phi + \xi\eta + \eta\phi + \xi\phi - \xi\eta\phi$$

$$N_4 = 1 - \eta - \xi - \phi$$

Eight Noded Hexahedral Element (HEXA – 8):

The linear hexahedral element or brick element as shown in fig, has eight corner nodes with isoparametric co – ordinates given by ξ , η & ϕ .



The lagrange shape function can be written as

$$N_i = 1/8(1+\xi\xi_i)(1+\eta\eta_i)(1+\phi\phi_i)$$

Where, $i = 1, 2, 3, \dots, 8$ and $(\xi_i, \eta_i \text{ \& } \phi_i)$ represents the coordinates of node i of the element in the natural coordinate system.

@ node 1: $\xi_1 = -1$, $\eta_1 = -1$, $\phi_1 = -1$

$$N_1 = 1/8(1+\xi\xi_1)(1+\eta\eta_1)(1+\phi\phi_1)$$

$$N_1 = 1/8(1+ \xi ^*-1)(1+\eta^*-1)(1+\phi^*-1)$$

$$\mathbf{N_1 = 1/8(1-\xi)(1-\eta)(1-\phi)}$$

@ node 2: $\xi_2 = 1$, $\eta_2 = -1$, $\phi_2 = -1$

$$N_2 = 1/8(1+\xi\xi_2)(1+\eta\eta_2)(1+\phi\phi_2)$$

$$N_2 = 1/8(1+ \xi ^*1)(1+\eta^*-1)(1+\phi^*-1)$$

$$\mathbf{N_2 = 1/8(1+\xi)(1-\eta)(1-\phi)}$$

@ node 3: $\xi_3 = 1$, $\eta_3 = 1$, $\phi_3 = -1$

$$N_3 = 1/8(1+\xi\xi_3)(1+\eta\eta_3)(1+\phi\phi_3)$$

$$N_3 = 1/8(1+ \xi ^*1)(1+\eta^*1)(1+\phi^*-1)$$

$$\mathbf{N_3 = 1/8(1+\xi)(1+\eta)(1-\phi)}$$

@ node 4: $\xi_4 = -1$, $\eta_4 = 1$, $\phi_4 = -1$

$$N_4 = 1/8(1+\xi\xi_4)(1+\eta\eta_4)(1+\phi\phi_4)$$

$$N_4 = 1/8(1+ \xi * -1)(1+\eta*1)(1+\phi*-1)$$

$$\mathbf{N_4 = 1/8(1-\xi)(1+\eta)(1-\phi)}$$

@ node 5: $\xi_5 = -1$, $\eta_5 = -1$, $\phi_5 = 1$

$$N_5 = 1/8(1+\xi\xi_5)(1+\eta\eta_5)(1+\phi\phi_5)$$

$$N_5 = 1/8(1+ \xi * -1)(1+\eta*-1)(1+\phi*1)$$

$$\mathbf{N_5 = 1/8(1-\xi)(1-\eta)(1+\phi)}$$

@ node 6: $\xi_6 = 1$, $\eta_6 = -1$, $\phi_6 = 1$

$$N_6 = 1/8(1+\xi\xi_6)(1+\eta\eta_6)(1+\phi\phi_6)$$

$$N_6 = 1/8(1+ \xi * 1)(1+\eta*-1)(1+\phi*1)$$

$$\mathbf{N_6 = 1/8(1+\xi)(1-\eta)(1+\phi)}$$

@ node 7: $\xi_7 = 1, \eta_7 = 1, \phi_7 = 1$

$$N_7 = 1/8(1+\xi\xi_7)(1+\eta\eta_7)(1+\phi\phi_7)$$

$$N_7 = 1/8(1+ \xi *1)(1+\eta*1)(1+\phi*1)$$

$$\mathbf{N_7 = 1/8(1+\xi)(1+\eta)(1+\phi)}$$

@ node 8: $\xi_8 = -1, \eta_8 = 1, \phi_8 = 1$

$$N_8 = 1/8(1+\xi\xi_8)(1+\eta\eta_8)(1+\phi\phi_8)$$

$$N_8 = 1/8(1+ \xi *-1)(1+\eta*1)(1+\phi*1)$$

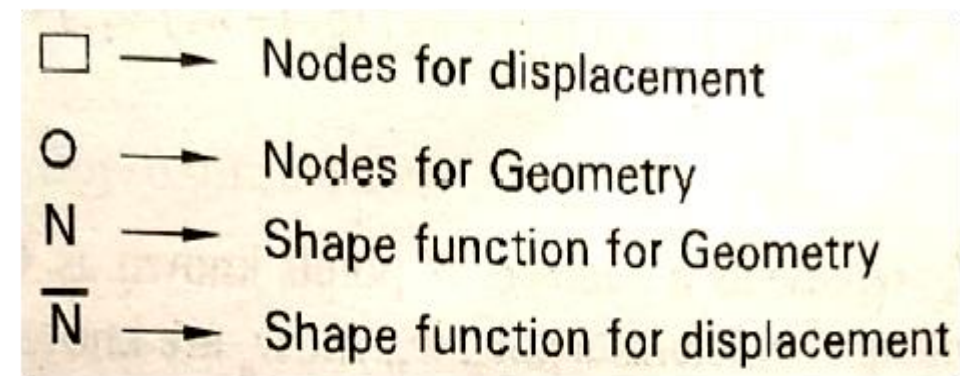
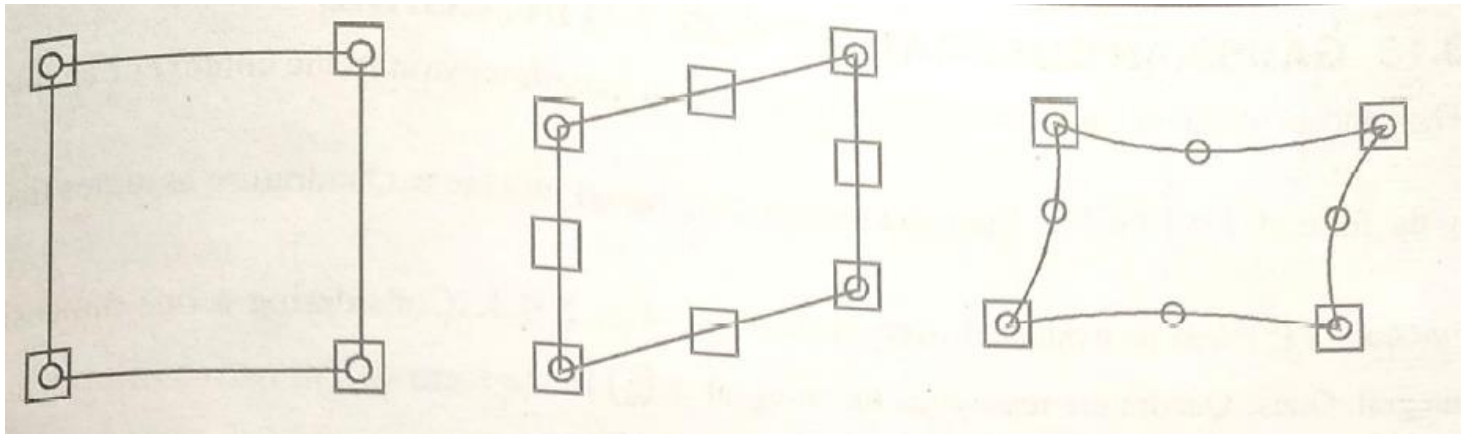
$$\mathbf{N_8 = 1/8(1-\xi)(1+\eta)(1+\phi)}$$

Iso Parametric, Sub Parametric and Super Parametric Element:

Iso Parametric Element:

If order of shape function used to define displacement are equal to the order of shape function used to define geometry, then such elements are known as ISO parametric element.

i.e, $\bar{N} = N$



Sub Parametric Element:

If order of shape function used to define displacement is greater than the order of shape function used to define geometry, then such elements are known as SUB parametric element.

i.e, $\overline{N} > N$

Super Parametric Element:

If order of shape function used to define displacement is less than the order of shape function used to define geometry, then such elements are known as SUPER parametric element.

i.e, $\overline{N} < N$

Numerical Integration:

Gauss Quadrature Formula:

An integral of the form $I = \int_{-1}^1 f(\xi) * d\xi$ can be integrated by gauss quadrature as

$$I = \int_{-1}^1 f(\xi) * d\xi = w_1 f(\xi) + w_2 f(\xi) + w_3 f(\xi) + \dots + w_n f(\xi)$$

where, $w_1, w_2, w_3, \dots, w_n$ are sampling weights

1,2,3,4,...n are sampling points

If 'n' samplings are used, this method can be used to integrate the polynomial of (2n-1) order.

One Dimensional Analysis:

A. One sampling point method:

Let us consider the gaussian quadrature formula with one sampling point i.e., $n = 1$

Then we have, $\int_{-1}^1 f(\xi) d\xi = w_1 f(\xi) \quad \text{-----} \quad 1$

One sampling point method can be used to integrate a polynomial of order $(2n-1)$

Therefore, $(2n-1) = 2(1) - 1 = 1$ (First order)

Assuming a polynomial of order one we have,

$$f(\xi) = a_0 + a_1 \xi \quad \text{-----} \quad 2$$

Substitute equation 2 in 1

$$\int_{-1}^1 (a_0 + a_1 \xi) d\xi = w_1 f(\xi)$$

$$\left[a_0 \xi + a_1 \frac{\xi^2}{2} \right]_{-1}^1 = w_1 f(\xi) \quad \left[a_0 + \frac{a_1}{2} \right] - \left[-a_0 + \frac{a_1}{2} \right] = w_1 f(\xi)$$

on simplification,

$$2a_0 = w_1 f(\xi)$$

For both the methods to give exact answer, the error should be zero

i.e., error = 0

$$2a_0 - w_1 f(\xi) = 0$$

$$2a_0 - w_1(a_0 + a_1 \xi) = 0$$

$$2a_0 - w_1 a_0 - w_1 a_1 \xi = 0$$

$$a_0(2 - w_1) - w_1 a_1 \xi = 0$$

But error is 0, only if $w_1 = 2$ and $\xi = 0$

Thus, one sampling point method is given by,

$$\int_{-1}^1 f(\xi) * d\xi = w_1 f(\xi) = 2 f(0)$$

One Dimensional Analysis:

B. Two sampling point method:

Let us consider the gaussian quadrature formula with one sampling point i.e., $n = 2$

Then we have, $\int_{-1}^1 f(\xi) d\xi = w_1 f(\xi) + w_2 f(\xi) \text{ ----- 1}$

One sampling point method can be used to integrate a polynomial of order $(2n-1)$

Therefore, $(2n-1) = 2(2) - 1 = 3$ (Third order)

Assuming a polynomial of order 3 we have,

$$f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \text{ ----- 2}$$

Substitute equation 2 in 1

$$\int_{-1}^1 (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) d\xi = w_1 f(\xi) + w_2 f(\xi)$$

$$\left[a_0\xi + a_1 \frac{\xi^2}{2} + a_2 \frac{\xi^3}{3} + a_3 \frac{\xi^4}{4} \right]_{-1}^1 = w_1 f(\xi) + w_2 f(\xi)$$

on simplification,

$$2a_0 + 2/3a_2 = w_1f(\xi) + w_2f(\xi)$$

For both the methods to give exact answer, the error should be zero

i.e., error = 0

$$2a_0 + 2/3a_2 - w_1f(\xi) - w_2f(\xi) = 0$$

$$2a_0 + 2/3a_2 - w_1(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) - w_2(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) = 0$$

$$a_0(2-w_1-w_2) - a_1(w_1\xi + w_2\xi) + a_2(2/3 - w_1\xi^2 - w_2\xi^2) - a_3(w_1\xi^3 + w_2\xi^3) = 0$$

Equating co-efficient of a_0, a_1, a_2, a_3

$$\text{We get, } 2 - w_1 - w_2 = 0 \Rightarrow w_1 + w_2 = 2$$

$$w_1\xi + w_2\xi = 0$$

$$w_1\xi^2 + w_2\xi^2 = 2/3$$

$$w_1\xi^3 + w_2\xi^3 = 0$$

These are non linear equations have a unique solution i.e., as follows

$$w_1 + w_2 = 2 \Rightarrow w_1 = w_2 = 1$$

$$w_1 \xi + w_2 \xi = 0 \Rightarrow \xi + \xi = 0 \Rightarrow \xi = -\xi$$

$$w_1 \xi^2 + w_2 \xi^2 = 2/3 \Rightarrow \xi^2 + \xi^2 = 2/3 \Rightarrow \xi = \frac{1}{\sqrt{3}} \text{ and } \xi = -\frac{1}{\sqrt{3}}$$

Thus, Two sampling point method is given by,

$$\int_{-1}^1 f(\xi) * d\xi = w_1 f(\xi) + w_2 f(\xi) = 1f\left(\frac{1}{\sqrt{3}}\right) + 1f\left(-\frac{1}{\sqrt{3}}\right)$$

Numerical on Gaussian quadrature:

1. Using gaussian quadrature formula, evaluate $I = \int_{-1}^1 \left[3e^{\xi} + \xi^2 + \frac{1}{\xi + 2} \right] d\xi$ by using one point and two point formula.

i. One point formula:

Sampling point, $n = 1$ and weights $w_1 = 2$ & $\xi_1 = 0$

$$I = w_1 f(\xi_1)$$

$$f(\xi_1) = 3e^{\xi_1} + \xi_1^2 + \frac{1}{\xi_1 + 2}$$

Put $\xi_1 = 0$

$$f(0) = 3e^0 + 0^2 + \frac{1}{0+2} = 3 + \frac{1}{2} = 3.5$$

Now,

$$I = w_1 f(\xi_1)$$

$$I = 2 * 3.5 = 7$$

ii. Two point formula:

Sampling point, $n = 2$ and weights $w_1 = w_2 = 1$ & $\xi_1 = -0.57735$, $\xi_2 = 0.57735$

$$I = w_1 f(\xi_1) + w_2 f(\xi_2)$$

$$f(\xi_1) = 3e^{\xi_1} + \xi_1^2 + \frac{1}{\xi_1 + 2}$$

$$f(-0.57735) = 3e^{-0.57735} + (-0.57735)^2 + \frac{1}{-0.57735 + 2}$$

$$f(-0.57735) = 1.6841 + 0.333 + 0.7029 = 2.72$$

Similarly,

$$f(0.57735) = 3e^{0.57735} + 0.57735^2 + \frac{1}{0.57735 + 2}$$

$$f(0.57735) = 5.3439 + 0.333 + 0.3880 = 6.065$$

Now,

$$I = w_1 f(\xi_1) + w_2 f(\xi_2) = 1(2.72) + 1(6.065) = 8.785$$

Two Dimensional Analysis/2D Integral:

The simplest two dimensional gauss rules are called product rules and are obtained by applying one dimensional rules to each independent variable in turn.

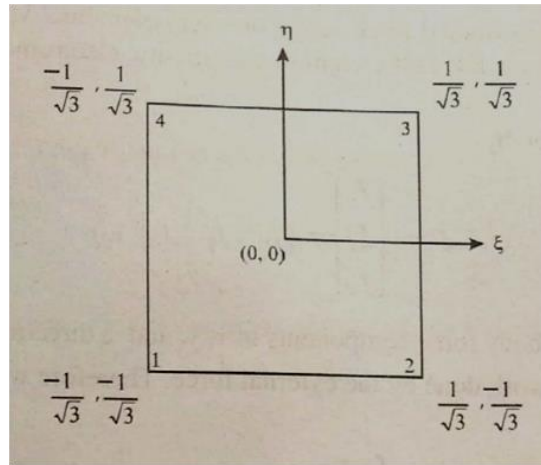
To be able to apply these rules we must first reduce the integrand to the canonical form

$$I = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta$$

$$I = \sum_{i=1}^n \sum_{j=1}^n w_i w_j f(\xi_i \eta_j)$$

$$\begin{aligned} I = & w_1 \{w_1 f(\xi_1 \eta_1) + w_2 f(\xi_1 \eta_2) + w_3 f(\xi_1 \eta_3) + \dots\} \\ & + w_2 \{w_1 f(\xi_2 \eta_1) + w_2 f(\xi_2 \eta_2) + w_3 f(\xi_2 \eta_3) + \dots\} \\ & + w_3 \{w_1 f(\xi_3 \eta_1) + w_2 f(\xi_3 \eta_2) + w_3 f(\xi_3 \eta_3) + \dots\} + \dots \end{aligned}$$

To find gauss point location ξ_i , η_i and weights w_i , consider a rectangular element in natural coordinates as shown in figure



$$I = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = w_1 f(\xi_1 \eta_1) + w_2 f(\xi_2 \eta_2) + w_3 f(\xi_3 \eta_3) + w_4 f(\xi_4 \eta_4)$$

Where, $w_1 = w_2 = w_3 = w_4 = 1$ and

$$(\xi_1 \eta_1) = \left(\frac{-1}{\sqrt{3}} \frac{-1}{\sqrt{3}} \right); (\xi_2 \eta_2) = \left(\frac{1}{\sqrt{3}} \frac{-1}{\sqrt{3}} \right)$$

$$(\xi_3 \eta_3) = \left(\frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \right); (\xi_4 \eta_4) = \left(\frac{-1}{\sqrt{3}} \frac{1}{\sqrt{3}} \right)$$

Force terms:

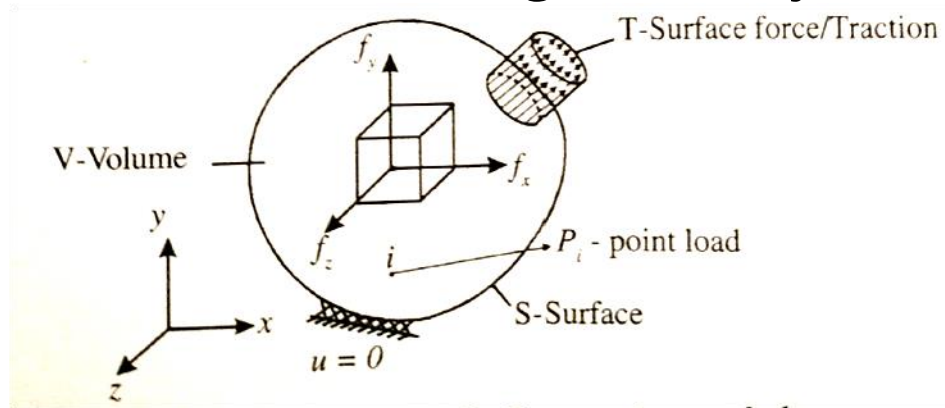
It is classified as follows,

1. Internal forces:

The internal forces are the reactive forces which are setup due to external forces and within elastic limits, the internal forces are numerically equal to the external forces.

2. External forces:

The state of stress and strain in a body arises due to external influences. The external forces acting on a body can be divided into three types as shown in figure



Body force:

Force acting on every elemental volume of the body.

Work potential due to body force:

Body force having 3 components F_x , F_y , F_z along x, y, z direction. $[f] = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$

Displacements due to body force, $[u] = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$

Where,

U, V, W are the displacements in x, y, z direction

$$[u]^T = [U \ V \ W]$$

Therefore,

$$\text{Work potential} = \int f U^T \cdot dv$$

Traction force:

Force acting on surface of the body.

Work potential due to traction force:

Traction force having 3 component T_x , T_y & T_z along x, y & z direction $[T] = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$

Displacements due to traction force, $[u] = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$

Where,

U, V, W are the displacements in x, y, z direction

$$[u]^T = [U \ V \ W]$$

Therefore,

$$\text{Work potential} = \int T U^T ds$$

Point load:

It is a force acting at particular point which causes displacement.

Work potential due to point load:

A point load $[P]$ is subjected to loading having components P_x , P_y & P_z along x, y & z direction

$$[P] = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

Displacements due to traction force, $[u] = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$

Where,

U, V, W are the displacements in x, y, z direction

$$[u]^T = [U \ V \ W]$$

Therefore,

$$\text{Work potential} = \Sigma P_i U^T$$



*Thank
you*